THE JANUARY MEETING OF THE CHICAGO SECTION.

THE twelfth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Friday and Saturday, January 2–3, 1903, the first session opening at 10:30 A. M. About thirty persons were in attendance, including the following members of the Society:

Professor Oskar Bolza, Professor D. F. Campbell, Professor E. W. Davis, Professor J. F. Downey, Professor Thomas F. Holgate, Dr. H. G. Keppel, Professor Kurt Laves, Professor H. Maschke, Professor E. H. Moore, Dr. F. R. Moulton, Professor H. B. Newson, Miss Ida M. Schottenfels, Professor J. B. Shaw, Dr. S. E. Slocum, Professor Henry S. White.

In the absence of the President or a Vice-President of the Society, Professor H. B. Newson was elected chairman of the Section. At the first session the Secretary reported that since the organization of the Section in 1897, one hundred and sixty-three papers had been read before it, by sixty-three different persons; and of these, seventy-two papers had been published. Later, during the meeting, the retiring President of the Society, Professor Moore, favored the Section with the presentation of his presidential address which had been delivered at the annual meeting in New York on December 30.

The following officers of the Section were elected for the ensuing year:

Secretary, Professor THOMAS F. HOLGATE.

Additional members of the programme committee, Professor E. B. SKINNER, Dr. S. E. SLOCUM.

The report of the committee appointed at the last Christmas meeting to devise a scheme of equivalent requirements for candidates proceeding to their second academic degree, with mathematics as their major subject, was taken up and discussed, final action being postponed until the April meeting. The report deals with the undergraduate programme which should be accepted as a basis for graduate work, and outlines in general terms the character of the work which should be demanded during the first year of graduate study. Copies of the report

are in the hands of the Secretary for distribution to members of the Society.

The following papers were read:

(1) Dr. SAUL EPSTEEN: "Determination of the group of rationality of a differential equation."

(2) Professor E. W. DAVIS: "A group in logic."

(3) Professor H. B. NEWSON: "On the generation of finite from infinitesimal transformations; a correction."

(4) Professor L. E. DICKSON: "The ternary orthogonal group in a general field."

(5) Professor L. E. DICKSON: "The group defined for a general field by the rotation groups."

(6) Professor A. S. HATHAWAY: "Vector analysis."

(7) Professor J. B. SHAW: "On nilpotent algebras."

(8) Professor D. F. CAMPBELL: "On homogeneous quadratic relations in the solution of a linear differential equation of the fourth order."

(9) Dr. S. E. SLOCUM: "Relation between real and complex groups with respect to their structure and continuity."

(10) Professor ARNOLD EMCH: "On the involution of stresses in a plane."

(11) Mr. R. E. WILSON: "Polar triangles of a conic and certain circumscribed quartic curves" (preliminary communication).

(12) Professor H. S. WHITE: "Orthogonal linear transformations and certain invariant systems of cones" (preliminary communication).

(13) Professor R. E. ALLARDICE: "On the envelope of the axes of similar conics through three fixed points."

Mr. Wilson was introduced by Professor White and Dr. Epsteen by Professor Maschke. In the absence of the authors, Professor Emch's paper and Professor Dickson's two papers were read by title; Professor Allardice's paper was read by Professor White, and Professor Hathaway's by Professor Shaw.

Professor Davis's paper appears in the present number of the BULLETIN. Professor Allardice's paper was published in the January number of the *Transactions*. Abstracts of the other papers are as follows:

1. Dr. Epsteen pointed out that when a special algebraic equation is given, its group can be theoretically determined 1° by constructing a $n!$ valued function and finding the Galois

resolvent, or 2° by applying the characteristic double property of the group, viz.: every rational function of the roots which remains unaltered by all the substitutions of G lies in R; and conversely. Both of these methods are generally impracticable, and in practice some device is resorted to in order to obtain the group. When a single rational function is known the following theorem has been found very useful: "If a rational function $\psi(x_1, \cdots, x_n)$ remains formally unaltered by the substitutions of a group G' and by no other substitutions, and if ψ equals a quantity lying in R and if the conjugates of ψ under G_n are all distinct, then the group of the given equation for the domain R is a subgroup of G' (cf. Dickson, Theory of algebraic equations, which is soon to appear, page 56).

The situation is exactly the same when we deal with linear differential equations. When a special linear differential equation is given, its group may be found: 1° by constructing the Picard resolvent, or 2° by applying the Picard-Vessiot characteristic double property of the group, viz.: every rational differential function (of a fundamental system) of the integrals which remains unaltered as a function of x, by the most general transformation of G lies in R and conversely. In this case these methods are even more impracticable than for the algebraic equations, and in order to determine the group a device should be resorted to. With this end in view the author has proved the following theorem: "If a rational differential function (of a fundamental system) of the integrals

$$\psi(y_1, \cdots, y_n;\ y_1', \cdots, y_n';\ \cdots;\ y_1^{(n-1)}, \cdots, y_n^{(n-1)})$$

remains formally unaltered by the transformations of a complex r-parameter linear homogeneous group G_r and if, when ψ is transformed by the most general linear homogeneous transformation, the resulting function contains $n^2 - r$ essential parameters, then the group of the given equation G is a subgroup of G_r."

4. The first paper by Professor Dickson is a study of the structure of the ternary orthogonal group in an arbitrary realm of rationality. The problem had been solved in two cases: for continuous fields by Weber (Algebra, II, second edition, pages 244–54); for finite fields by the writer (Linear Groups, page 164). The group of all ternary orthogonal transformations of

determinant unity in a field F is designated $O(3, F)$. If $i = \sqrt{-1}$ belongs to F, $O(3, F)$ is simply isomorphic with the group of all linear fractional transformations in F. The latter has as an invariant subgroup the simple group of all linear fractional transformations in F of determinant unity. Simple generators of $O(3, F)$ are given. If i extends F to a larger field $F(i)$, then $O(3, F)$ is simply isomorphic with the group of all transformations

$$z' = \frac{\alpha z + \beta}{-\sigma\bar{\beta}z + \sigma\bar{\alpha}} \qquad \begin{pmatrix} \alpha, \beta \text{ in } F(i), \\ \sigma\bar{\sigma} = 1. \end{pmatrix}$$

The transformations with $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$ form a subgroup, the fractional binary hyperorthogonal group (compare Linear Groups, page 132). Certain theorems on this group are established.

5. The second paper by Professor Dickson is a contribution to the theory of group matrices and group determinants, investigated by Frobenius and further developed by Burnside for continuous fields and by the writer for an arbitrary field. The paper employs only very elementary methods and is quite independent of the papers cited. For the cyclic group g_k, the dihedron group g_{2k}, the alternating and symmetric groups on four letters, and the alternating group on five letters, the group matrix is reduced to a simple canonical form of the type shown to exist by Frobenius, the reducing transformation being exhibited explicitly. In particular, the irreducible factors of the group determinant are given. Both papers (printed October 1) are to appear in the *University of Chicago Decennial Publications*, Volume 9.

6. Professor Hathaway contends that if a vector analysis short of quaternions is desired, then all attempts at bringing in conceptions of products of vectors should be avoided as necessarily incomplete and uselessly metaphysical. Any quantity whose value depends upon and is determined by the values of given arguments $\mathbf{xyz}$ is simply a function of those arguments, and the ordinary form of notation $f\mathbf{xyz}$ should suffice.

The requisite analysis is afforded by the linear functions, defined with respect to any argument $\mathbf{x}$, by

$$f(\mathbf{x} + \mathbf{x}')\mathbf{yz} = f\mathbf{xyz} + f\mathbf{x}'\mathbf{yz}.$$

In consequence, a scalar factor of an argument of a linear function is simply a factor of the function; and we have general expansions of the type

$$f(\mathbf{x}+\mathbf{x}')(\mathbf{y}+\mathbf{y}')\mathbf{z} = f\mathbf{xyz} + f\mathbf{xy}'\mathbf{z} + f\mathbf{x}'\mathbf{yz} + f\mathbf{x}'\mathbf{y}'\mathbf{z}.$$

The similarity of this to distributive multiplication is evident; and any symmetric or alternate properties of the function concerned impose corresponding commutative laws. Only three linear functions (a *scalar* and *vector* alternate, and a *scalar* symmetric) need be permanently defined:

$S\mathbf{abc}$ = volume of parallelepiped whose edges are **abc** in order.

$V\mathbf{ab}$ = vector of the area of the parallelogram whose edges are **ab** in order.

$S\mathbf{ab}$ = volume of parallelepiped of edge **a** and vector base area **b**.

7. In Professor Shaw's paper it was shown that the general form of any number of a nilpotent algebra is

$$\phi = \sum a_{ijk}\lambda_{ijk},$$

where $\lambda_{ijk}\lambda_{j'k'} = \lambda_{ik+k'}$, $\lambda_{ijk}\lambda_{j'k'} = 0 (j \neq j')$, and in any case λ_{rst} does not exist unless t is positive or zero, and $\mu_r - \mu_s - 1 < t < \mu_r$. The numbers μ_r, μ_s, etc., are certain "multiplicities" in the expressions. The units of the algebra may be expressed in the form

$$\phi_1,\ \phi_2,\ \cdots,\ \phi_m,\ \phi_{m+1},\ \phi_1\phi_{m+1},\ \phi_1\phi_{m+1}^2,\ \cdots$$

$$\phi_2\phi_{m+1},\ \phi_2\phi_{m+1}^2,\ \cdots$$

where .....

$$\phi_r\phi_s = a\phi_{s+1} + b\phi_{s+2} + \cdots + f\phi_{m+1} + g\phi_1\phi_{m+1} + \cdots$$

Certain typical cases are given as examples.

8. Professor Campbell's paper is in abstract as follows: Let

$$\sum_{i=1}^{i=4}\sum_{j=1}^{j=4} a_{ij}y_iy_j \qquad (a_{ij} = a_{ji})$$

be a homogeneous quadratic form in the solutions y_1, y_2, y_3, y_4

of the linear differential equation $y^{iv} + 6p_2y'' + 4p_3y' + p_4y = 0$. It can be written symbolically as

$$\left(\sum_{i=1}^{i=4} \alpha_i y_i\right)^2.$$

If successive derivatives be taken of this form and use be made of the fact that $y_i^{iv} = -6p_2y_i'' - 4p_3y_i' - p_4y_i$ $(i = 1, 2, 3, 4)$, any derivative will be composed of terms, each of which consists of a constant or a rational integral function of the coefficients of the given differential equation, and their derivatives, multiplied by the symbolic factors

$$\left(\sum_{i=1}^{i=4} \alpha_i y_i\right)^{r_0}\left(\sum_{i=1}^{i=4} \alpha_i y_i'\right)^{r_1}\left(\sum_{i=1}^{i=4} \alpha_i y_i''\right)^{r_2}\left(\sum_{i=1}^{i=4} \alpha_i y_i'''\right)^{r_3}$$

where $r_0 + r_1 + r_2 + r_3 = 2$. These factors may be kept in the order

$$\left(\sum_{i=1}^{i=4} \alpha_i y_i^{(j)}\right) \qquad (j = 1, 2, 3),$$

and indicated by the exponents $[r_1, r_2, r_3]$. Then the derivative of the symbolic part of each term of any derivative will be given by the formula

$$\frac{d\,[r_1r_2r_3]}{dx} = r_1[r_1 - 1, r_2 + 1, r_3] + r_2[r_1, r_2 - 1, r_3 + 1]$$

$$+ (2 - r_1 - r_2 - r_3)\,[r_1 + 1, r_2, r_3] - 6p_2r_3[r_1, r_2 + 1, r_3 - 1]$$
$$- 4p_3r_3[r_1 + 1, r_2, r_3 - 1] - p_4r_3[r_1, r_2, r_3 - 1].$$

The following theorems are used in the investigations:

First. If there are g functions U_i, $i = 1, 2, \cdots g$ of an independent variable among which ν and only ν linearly independent relations exist, then

$$u = \sum_{i=1}^{i=g} a_i U_i$$

a_i arbitrary constants, is the general solution of a linear differential equation of order $g - \nu$.

Second. The necessary and sufficient condition that the n functions of x, U_i $(i = 1, 2, \cdots, n)$ are linearly independent is that the determinant

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix}$$

vanishes.

Third. In any homogeneous quadratic relation $u = [000]$ that may exist in the solutions of the given differential equation, the expression $[200]$ cannot vanish.

Fourth. The relations $[000] = 0$, $[\overline{000}] = 0$, $\cdots$ in the solutions of the given differential equation are linearly independent if $[200]$, $[\overline{200}]$, $\cdots$ are linearly independent.

By successive differentiation of $u = [000]$ there result ten equations in the expressions $[r_1r_2r_3]$ whose coefficients are constant or rational integral functions of the coefficients of the differential equation and their derivatives, and whose right hand members are $u, u', u'', \cdots, u^{ix}$. Arrange the terms so that the expressions appear in the order $[000]$, $[100]$, $[010]$, $[200]$, $[110]$, $[101]$, $[020]$, $[011]$, $[001]$, $[002]$. Form the determinant Δ of the coefficients. Denote any determinant formed from the matrix derived from Δ by omitting the last n rows, $n = 1, 2, 3$, by the numbers indicating the columns chosen for the determinant. It is shown that:

The necessary and sufficient conditions that one and only one homogeneous quadratic relation exists in the solutions of the given differential equation are that Δ vanishes and (1235678910) does not vanish. The necessary and sufficient conditions that two and only two linearly independent homogeneous quadratic relations exist in the solutions of the given differential equation are that (1235678910) vanishes and not all (123478910), (123578910) and (123678910) vanish. The necessary and sufficient conditions that three linearly independent homogeneous quadratic relations exist in the solutions of the given differential equation are that (123478910), (123578910) and (123678910) vanish. There cannot be more than three linearly independent homogeneous quadratic relations in the solutions of the given differential equation.

9. The first part of Dr. Slocum's paper finds the form of the infinitesimal transformation U_a by which any given finite transformation T_a of a group with continuous parameters is generated when the equations defining this finite transformation are in their *non-canonical* form. For certain values of the parameters the transformation U_a may not be infinitesimal, in which case the corresponding finite transformation T_a is not generated by an infinitesimal transformation of the group.

Associated with each r-parameter structure is a certain determinant the constituents of which are functions of the r-parameters. If any system of values of the parameters can be found for which this determinant vanishes, the parameter group belonging to that type of structure is discontinuous.

A group may be continuous as a real group but discontinuous as a complex group. Consequently the idea, as developed by Lie, of the relation between the transitivity of the real and complex groups which have the same symbols of infinitesimal transformation, requires modification.

There being more types of structure possible for real groups than for complex groups, if we have given two structures which are of the same type, A, for complex groups, but which constitute distinct types, B and C, for real groups, it is shown by means of examples that one of the following cases may occur with respect to the continuity of the real groups of types B and C: 1° All real groups of both types B and C are continuous. 2° All real groups of both types B and C are discontinuous. 3° All real groups of one type, B, are continuous, and one or more, but not all, real groups of the other type, C, are discontinuous. 4° All real groups of one type, B, are continuous, and all real groups of the other type, C, are discontinuous.

10. Through Culmann's investigations in graphical statics it is known that the directions corresponding to the sections and the resultant stresses upon these sections at every point of a thin strained plate, form an involutoric pencil. This involution is hyperbolic or elliptic according as the plate is either subject to tensions and compressions, or to tensions or compressions only. Professor Emch briefly discusses these cases, establishes the equation of the stress ellipse and the differential equation of the tension and compression curves, and finally applies the results to the linear infinitesimal deformation of the plane.

11. Following a plan outlined by Professor White, in a paper read before the Society in September, 1902, an attempt is made by Mr. Wilson to investigate the number of independent relations that must exist among the coefficients in order that a quartic curve shall contain an infinity of triangles self-conjugate with respect to a given conic.

The binary quartic and conic were taken in the form

$$f_x^4 = A_x^2 a_x^2 \equiv 0, \quad \phi_x^2 = \alpha_x^2 \equiv 0.$$

The condition that two roots of the quartic shall separate two roots of the conic harmonically is

$$\{(a\alpha)^2(A\alpha)^2\}\{(AB)^2(ab)^2(\alpha\beta)^4 - (\alpha\beta)^2(A\alpha)^4 + 4(\alpha\beta)^2(a\alpha)^2(A\alpha)^2(A\alpha)^2 - 4(aa)^4(A\alpha)^4\} = 0$$

(see Clebsch, Binäre Formen.) Reducing and expressing in terms of invariants of $f_x^4 = 0$ and $\phi_x^2 = 0$, this becomes

$$\bar{A}^3 - \tfrac{1}{3}jA^3 - \tfrac{1}{2}iA\Delta^2 \equiv 0,$$

where i and j are invariants of $f_x^4 = 0$ alone, Δ of $\phi_x^2 = 0$, and $\bar{A}$ a simultaneous invariant of f and ϕ.

By bordering, the condition is obtained that a line $\mu_x = 0$ (ternary) shall satisfy the harmonic division. By considering the locus of the pole of $\mu_x = 0$, a 12th-ic is obtained which breaks down to a sextic $C_x \equiv 0$. The general sextic contains 28 coefficients, and hence the maximum number of conditions is 28.

The curves

$$f_x^4 = 4ax_2^3x_3 + 4bx_3^3x_1 + 4cx_1^3x_2 = 0, \quad \phi_x^2 = x_1^2 + x_2^2 + x_3^2 = 0$$

have been examined and show that the quartic above cannot contain infinitely many inscribed triangles self-conjugate with respect to a conic which has the triangle $x_1x_2x_3 = 0$ as a self-conjugate triangle.

The forms

$$f_x^4 = 6\sum a_1x_1^2x_2^2 + 12\sum b_1x_1^2x_2x_3 = 0, \quad \phi_x^2 = x_1^2 + x_2^2 + x_3^2 = 0$$

have also been examined. Although the result is not in complete form, it has disclosed properties that promise to lend interest to investigations other than the one in hand.

12. The formulas of Euler (and Rodrigues), which give rationally in terms of three parameters the nine coefficients of an orthogonal transformation in space, are available for expressing the vertices of any polar triangle of the conic

$$x_1^2 + x_2^2 + x_3^2 = 0.$$

This gives a means of discussing curves of the second and third orders, at least, which contain infinitely many inscribed polar triangles of a conic. Professor White's preliminary communication exhibited the method as applied to conics. The further question was raised, what sort of curve is the locus of points whose coördinates are the eulerian parameters of polar triangles inscribed in a single conic, or of orthogonal transformations which rotate the axes through the surface of an orthogonal quadric cone.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILLINOIS.

SOME GROUPS IN LOGIC.

BY PROFESSOR E. W. DAVIS.

(Read before the Chicago Section of the American Mathematical Society, January 2, 1903.)

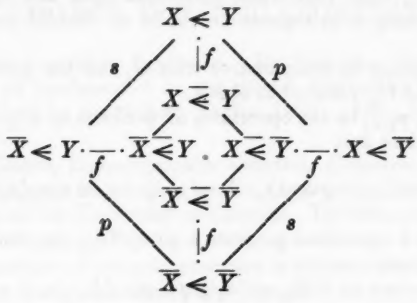
DE MORGAN has pointed out * that his eight forms of proposition identical with the *A*, *E*, *I*, *O*, and their contranominals of the older logic, can be derived from any one by the three operations of reversing the subject, reversing the predicate, denying the copula. If, in fact, we denote the operations in question by *s*, *p*, and *f* respectively, we have

$$Ap = E, \quad Af = O, \quad Afp = I;$$

while *sp* changes any proposition to its contranominal $X \prec Y$ to $Y \prec X$; or, what is the same thing to $\overline{X} \prec \overline{Y}$. Here $\prec$ is the sign of implication and the bar written over a letter or sym-

* Formal Logic, p. 63 et seq.

bol reverses or denies the meaning of the same. All the relations can be simply expressed by a diagram



This scheme differs from De Morgan's in that he assumes always the existence of both a subject and its contrary, whereas our assumption, following Mr. C. S. Peirce,* is that $X \leqslant Y$ means 'not X or else Y,' while $X \overline{\leqslant} Y$ means 'X but not Y.'

There is also a group of order 32 which operating upon any one of 24 forms of valid syllogism will produce the rest, together with 8 non-valid syllogisms. To see this, start with *Barbara*

$$(X \leqslant Y)(Y \leqslant Z) \leqslant (X \leqslant Z),$$

or say

$$A_1 A_2 \leqslant A_3.$$

Let now a_1 mean the operation of performing *sfp* upon A_2 and A_3, with similar significations for a_2 and a_3. Then, B standing for *Barbara*, Ba_1 and Ba_2, are valid, while Ba_3 is not.† Moreover

$$G_4 = \{1, a_1, a_2, a_3\}$$

is a 4-group. Were B, instead of *Barbara*, any valid syllogism it would still be true that of Ba_1, Ba_2, and Ba_3, two would be valid while one was invalid. If B were invalid the operations a might lead to none that were valid, could at most lead to one that was.

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* *Amer. Journ. of Math.*, vol. 3, p. 18.

† Conversion of the propositions with negative copula reveals the fact that B, Ba_1, Ba_2, Ba_3, are in the four figures of the older logic.

Suppose x to be the operation of interchanging X and $\overline{X}$, with like significations for y and z. These 3 operations generate a group G_8, such that when performed upon any syllogism B the resulting 8 syllogisms are valid or invalid according as B is.

The group is commutative with G_4 and the product of the two is the G_{32} referred to above.

Let s_i, p_i, f_i be the operations of performing s, p, f upon A_i. It is evident that

$$x = s_1 s_2, \quad y = p_1 s_2, \quad z = p_2 p_2, \quad a_1 = s_2 p_2 f_2 s_3 p_3 f_3.$$

These 9 operations generate a group G_{512}, the product of G_{32} by the group

$$G_{16} = \{x_1 s_1,\ p_1,\ p_2,\ f_1\}.$$

The new syllogisms got by operating with this upon the 32 syllogisms derived from *Barbara* above are all invalid.

The operation called conversion interchanges the subject and predicate of a proposition, at the same time reversing each. The meaning is, of course, left unchanged. Immaterial to the argument likewise is it whether the order of the propositions be $A_1 A_2$ or $A_2 A_1$. Let the operation of converting A_i be denoted by c_i while the interchange of A_1 and A_2 is denoted by t. Then $\{c_1, c_2, c_3, t\}$ is of order 16. We call it H_{16}. Finally

$$G_{512} \times H_{16} = G_{8192},$$

whereof

$$G_{32} \times H_{16} = H_{512}$$

contains the operations which performed upon *Barbara* lead to all the valid syllogisms of BG_{8192} taking account of conversion and the order of the premises, viz., to 384.

The same sort of group building can of course be applied to more complicated sets of statements, and is a very simple consequence of the binary character of deductive logic.

UNIVERSITY OF NEBRASKA,
January 11, 1903.

CESÀRO'S INTRINSIC GEOMETRY.

Lezioni di Geometria Intrinseca. By ERNESTO CESÀRO, professor of mathematics at the Royal University of Naples. Published by the author, Naples, 1896. 8vo., 264 pp., 48 figures.

Ernesto Cesàro, Vorlesungen über natürliche Geometrie. Authorized German translation by Dr. GERHARD KOWALEWSKI, docent at the University of Leipzig. Teubner, 1901. 8vo., vi + 341 pp., 48 figures.

THE subject of intrinsic geometry is almost unknown among American mathematicians, yet it has quite an extensive literature, which has been recently collected and published by Wölffing.* The name Cesàro occurs more frequently than any other in this report.

The work here under consideration contains considerable new matter, but its principal mission is to provide a systematic treatment of the subject from the beginning. The success of the effort is attested to by the fact that the work received honorable mention at the awarding of the Lobachevsky prize in 1897.

The German translation differs but little from the Italian original, except that it corrects a considerable number of typographical and other errors, and embodies in the text a number of notes appended to the original. The further changes will be mentioned later.

The work is divided into seventeen chapters, of which eight treat of plane geometry, two of twisted curves, three of surfaces, one of line congruences, and finally the last three of three-dimensional space, of curves in hyperspace, and of hyperspaces, respectively. There is also an appendix giving a generalization of Grassmann's numbers, a discussion of the equilibrium of flexible but non-extensible wires, and the equations of elasticity in hyperspace.

The physical improvements in the German edition, the spaced paragraphing, the full index (wholly lacking in the original)

* Bericht über den gegenwärtigen Stand der Lehre von den natürlichen Coördinaten, *Bibliotheca Mathematica*, Ser. 3, vol. 1, pp. 142–159.

and the more distinctive type used for printing the illustrative examples, are all of great assistance to the reader.

The subject matter proper of the book is not preceded by an introduction, and no references are given, both of which omissions greatly limit the usefulness of the work. The lack has now to a large extent been supplied, so far as the plane is concerned, by Wölffing's report mentioned above.

In the first five chapters a curve is defined by a relation between s, the length of arc measured from any point of the curve and ρ, the radius of curvature. Frequently rectangular coördinates (u, v) referred to the tangent and normal are used. The first chapter gives a good sample of the methods employed, by deriving a large number of interesting properties of certain curves in a brief and strikingly elegant manner.

Let $M(o, o)$ and $M'(u, v)$ be two points on a curve. Since $\sqrt{u^2+v^2} < \delta s < u+v$, and $\lim v/u = 0$, it follows that $\lim \delta s/u = 1$. If MM' makes an angle $\delta\phi$ with the tangent at M, $1/\rho = \lim \delta\phi/\delta s$, $\lim v/u^2 = 1/\rho$. These ideas suffice to obtain the forms of the circle $\rho = a$, the catenary $\rho = a + s^2/a$, and the catenary of uniform resistance $\rho = \frac{1}{2}a(e^{-s/a} + e^{-s/a})$. It is to be observed that the origin and axes are variable and are only intermediate steps in the analysis.

Inflexions and cusps appear according as

$$\lim \frac{v}{u^{2-n}} = \frac{1}{(2-n)(1-n)} \lim \frac{s^n}{\rho}$$

is less or greater than zero. Asymptotes are characterized by $s = \infty$ when $1/\rho = 0$ and $\lim \phi$ is finite. Asymptotic circles result when $\lim \rho = a$, as $\phi = \infty$, and asymptotic points appear when $a = 0$. The study of these points closes the chapter.

The examples discuss the involute of the circle $\rho^2 = 2as$, the form of the tractrix, the proof that the segment of the tangent cut off by the asymptote is constant, and a simple construction for the center of curvature of any point. The German edition compares the forms of the two catenaries, and shows that the catenary of uniform resistance has two parallel asymptotes. The cycloids are defined by the equation

$$\frac{s^2}{a^2} + \frac{\rho^2}{b^2} = 1;$$

the hypocycloid or epicycloid is defined, according as a is

greater or less than b, and the common cycloid is the intermediate form, when $a = b$. The whole discussion, including eight figures, occupies less than two pages. Its directness must be admitted, but the periodic form of the curve is not established until an auxiliary variable is employed. From the form of the equation, no direct knowledge of the form of the curve beyond a cusp is given. The author makes no further mention of the matter, but ignores the fact that he departs from his otherwise consistent procedure to establish the result. This departure is, however, unnecessary if account be taken of the change of sign which the arc undergoes when passing a cusp. The same scheme will apply to all the curves which have cusps, whether periodic or not. This excellent list of examples closes with a discussion of the curves

$$\rho = ks^n,$$

which include the involute of the circle ($n = \frac{1}{2}$), the logarithmic spiral ($n = 1$) and the double spiral ($n = 2$) which Cesàro calls the clotoid.*

The second chapter discusses the fundamental formulas

$$\frac{dx}{ds} = \frac{y}{\rho} - 1, \quad \frac{dy}{ds} = -\frac{x}{\rho},$$

which are necessary and sufficient for the immovability of the point (x, y) not on the curve. These formulas are all that are needed for expressing the equation of any locus which is invariantly connected with the curve. Here envelopes, evolutes, parallel curves are discussed and a rich collection of examples is given which illustrate how to derive the intrinsic equations of curves defined as a geometric locus. The method is rather too abstract, and not sufficient use is made of coördinate axes. For example, it is proved that the evolute of a cycloid is a similar cycloid, but an indirect procedure is necessary to determine the relative positions of the two curves. The same fact is still more evident in case of the hypocycloid and epi-

* For a further discussion of the clotoid, and the literature upon it, see Loria's recent work, Le curve piane algebriche e transcendenti, teoria e storia; German by Schütte, Specielle algebraische und transcendente Curven in der Ebene, Theorie und Geschichte, Leipzig, Teubner, 1901, p. 457. It would be interesting to compare the various double spirals with the stereographic projection of the logarithmic spiral.

cycloid. The relation between ρ_n, the radius of curvature of the nth evolute, is shown to satisfy the equation

$$\rho_n = \rho \frac{d\rho_{n-1}}{ds}.$$

The third chapter is devoted to the discussion of particular curves. It begins with a review of five pages on the ordinary properties of conics, derived by the well known methods, after which the equation between s and ρ is expressed by means of an elliptic integral of the second kind. The equation of the cassinian oval is expressed as a hyperelliptic integral of the first kind, for which $p = 3$. A family of curves is also discussed for which Wölffing has suggested the name Cesàro curves. They are defined by the following property: the radius of curvature at any point is proportional to the segment on the normal between the point of tangency and its intersection with the polar of the point of tangency with regard to a fixed circle. When the circle reduces to a point the curve becomes a sinuous spiral. When the circle becomes a straight line, the locus is called a Ribaucour curve. Although various curious properties are derived, the method does not seem adapted to the study of the general topology of the curves; no figures are given except of the well known forms. The German edition has added one more paragraph [§ 46] to this chapter, dealing with the investigation of those curves whose osculating circles, dilated from the point of contact, all belong to a linear complex of circles. The locus of centers is a Cesàro curve.

In the fourth and fifth chapters, contact and osculation, and roulettes, the power of the method again becomes apparent. The ordinary properties of contact, curvature, successive evolutes, maximum and minimum curvature, etc., are very skilfully derived. In the chapter on roulettes a more detailed study of the cycloid is given, and a number of elegant relations between various Cesàro curves are derived.

Thus far the procedure has been successive; a small number of formulas was first derived and a large number of consequences drawn from them. The remaining chapters (except the ninth and tenth) use many auxiliary coördinates.

Chapter VI deals with barycentric coördinates. It would grace any work on modern geometry, and the analysis seems to lend itself readily to the method in question. Centers and lines

of gravity are discussed at length and applied to a large number of curves. Chapter VII, barycentric analysis, is a direct continuation of the preceding. Equations of a line, pair of lines, conic section, tangent, normal, pole and polar, are derived in much the same manner as in the ordinary texts on trilinear coördinates. The chapter closes with a well written paragraph on the symmetric triangular curves and the anharmonic curves. The lack of references here seems particularly unfortunate, for it is of considerable value to compare the treatment here given with that of Laguerre, Halphen, Klein and Lie.

The last chapter in the part on plane geometry is devoted to pencils of plane curves. Let u be a function of position in the plane. If u changes by the amount du, when the point moves through the distance ds,

$$\frac{du}{ds} = \cos\omega \frac{\partial u}{\partial s_1} + \sin\omega \frac{\partial u}{\partial s_2},$$

in which ds_1, ds_2, represent orthogonal components of ds. The expression

$$\Delta u = \left(\frac{\partial u}{\partial s_1}\right)^2 + \left(\frac{\partial u}{\partial s_2}\right)^2$$

is called the first differential parameter. If the pencil $u \equiv \kappa$ consists of parallel curves, Δu is a function of u alone. The second differential parameter

$$\Delta^2 u = \left(\frac{\partial}{\partial s_1} + \frac{\partial \omega}{\partial s_2}\right)\frac{\partial u}{\partial s_1} + \left(\frac{\partial}{\partial s_2} + \frac{\partial \omega}{\partial s_1}\right)\frac{\partial u}{\partial s_2}$$

vanishes for an isothermal system, provided u is the isometric parameter. These considerations furnish a good introduction to curvilinear coördinates, their fundamental formulas, and the conditions for integrability which follow. Then are added the relations among curvilinear coördinates, which were first derived by Lamé and by Bonnet, but which are usually found in works on the theory of surfaces.

The remainder of the book is devoted to geometry of more than two dimensions. Chapter IX treats of twisted curves and ruled surfaces. In the first part pure intrinsic geometry again becomes prominent, but the digression on the straight line

and the application of the results to scrolls, developables, and associated surfaces makes a wide departure from the otherwise consistent development. One has the feeling that the arrangement is far from natural in the sense of each result suggesting the following one, and the deductions frequently leave several steps unnoticed or assumed. The proof of the theorem that a developable must be the envelope of tangents of a twisted curve is direct and elegant. In establishing Chasles's correlation between tangent planes and points of tangency on a scroll the two lines g, $g + \delta g$ are confused; the tangent plane should contain the generator through the point of tangency. The next chapter discusses some particular twisted curves. Spherical curves, helices, and conical loxodromes are treated at some length and a number of miscellaneous examples are added. A discussion of the Bertrand curves, defined by a linear relation between curvature and torsion, is followed by a study of the curves satisfying the quadratic relation

$$\frac{A}{\rho^2} + \frac{B}{r^2} + \frac{C}{\rho r} = \frac{P}{\rho} + \frac{Q}{r}.$$

Although the various properties mentioned are all derived, the method of deduction is so briefly outlined and explanations are so sparingly used that the text would prove to be almost impossible reading to any one not fairly familiar with the results of the ordinay theory.

Chapter XI, general theory of surfaces, is largely geometric in character and not so materially different from the treatment in the ordinary books, except that it is remarkably brief and direct. Most of us are not satisfied with these considerations as proofs, but they are of immense value when accompanied by the lively geometric interpretation. There are no natural coördinates here as in the case of curves, but both this treatment and the ordinary one would be greatly improved by drawing from each other. One interesting theorem is the determination of the curvature in the tangent plane in a hyperbolic point (theorem of Beltrami), as the theorems of Euler and Meusnier do not apply to this case. The result shows that the curvature of the section made by the tangent plane is two thirds that of the asymptotic line which it touches.

The next statement seems trivial; it says that two curves may touch each other and have the same osculating plane at

the point of contact without having the same curvature at that point.

Clairaut's theorem regarding geodesics on surfaces of revolution is very neatly proved.

The German edition adds a theorem regarding the curvature of a normal section of the circumscribing cylinder of a surface (curve of apparent contour).

A good discussion of surfaces of constant curvature is given, and figures of the various types are discussed. Asymptotic lines on the pseudosphere are derived and the form of the lines obtained. Applicability, minimum surfaces and the Weingarten surfaces are briefly treated.

These two chapters are very valuable when read in connection with the older treatment of the subject; but the outlook is by no means hopeful that the later method will prove sufficiently powerful to secure many new results from the theory of surfaces. The chapter closes with rather a full discussion of surfaces of the second order.

Chapter XIII, infinitesimal deformations of a surface, discusses the possible deformations and the consequent changes of curvature. It is skilfully proved that the mean curvature of a mimimum surface must be everywhere zero, and that the curvature of any surface is not changed by deformation.

The next chapter is devoted to rectilinear congruences. It is a rapid review of the work of Kummer, which is so well presented in the work of Bianchi. A certain arbitrariness is here apparent and a strong suspicion is unavoidable that all the results were known beforehand; moreover, one feels that the present method is forced somewhat in reproducing those results. The results of Sturm on the configuration of the normals of a surface near a given point are added to the older theorems of Hamilton and Kummer, and some new relations among the derivatives are obtained analogous to those of Codazzi for surfaces. By means of these there is given a noteworthy proof of the theorem that the mean envelope of an isotropic congruence is a minimum surface. The second half of this chapter is considerably altered in the German edition.

Chapter XV treats of three dimensional space. The discussion is very similar to the introduction of curvilinear coördinates in the plane. All the tangent lines to a surface at a particular point lie in a plane. In any triply orthogonal system the curves of intersection are lines of curvature. Every

pencil of parallel surfaces belongs to a triple orthogonal system. Several paragraphs are devoted to the derivation of formulas analogous to Codazzi's.

Thus far the space is considered linear, so that the element of arc, referred to the tangents of the three orthogonal lines of curvature, is expressed in the form

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2.$$

If the space is not linear but has a constant curvature $1/R$, it is shown that the element of arc is defined by

$$ds = \frac{\sqrt{dq_1^2 + dq_2^2 + dq_3^2}}{1 + \frac{1}{4R^2}(q_1^2 + q_2^2 + q_3^2)}.$$

Apart from giving the impression of being rather overloaded, this chapter is very interesting and the power of the methods is made particularly apparent in all of the geometric part of it.

Chapter XVI treats of curves in space of n dimensions. The various normals are first defined, the conditions for orthogonality deduced, and then the analagons to Frenet's formulas are derived, which are called the fundamental formulas for the intrinsic analysis of curves in hyperspace.

In the German edition an article is added which is interesting because it discovers a fundamental difference in the nature of spaces having an even number of dimensions from those having an odd number. The locus of a point having first, second, $\cdots$, nth curvatures all constant is spherical or helical, according as n is even or odd.

The last few pages of this chapter are devoted to barycentric analysis of hyperspace, quite analogous to the projective system of Professor Stringham.

The last chapter preceding the appendix is on hyperspaces. The generalizations of the theorems of Euler and Dupin are first derived, then the formulas of Codazzi and the criteria for infinitesimal deformation. The volume closes with the remarkable theorem of Beez that a non-extensible hypersurface cannot be deformed.

The book represents an important and a beautiful work, and the reader's admiration is constantly excited for the author's versatility and power of assimilating the existing literature.

Two general defects seem to be patent, one in presentation, one in range. It is the author's aim to emphasize the force and directness of the analysis by having the results appear after a minimum of intermediate work, but this work has frequently been so abbreviated that the reading is at times extremely difficult. Again, the author was naturally ambitious to have his method apply to as large a range of subjects as possible, but its applicability seems but questionably successful when applied to projective properties.

For the differential analysis of curves and surfaces the work of Cesàro is certainly a powerful instrument and can be used with profit by every student of the subject.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
December, 1902.

GAUSS'S COLLECTED WORKS.

Carl Friedrich Gauss Werke. Achter Band. Herausgegeben von der königlichen Gesellschaft der Wissenschaften zu Göttingen. In Commission bei B. G. Teubner in Leipzig, 1900. 4to., 458 pp.

WITH the death of Ernst Schering, the venerable editor of the first six volumes of Gauss's collected works, it became incumbent on younger hands to continue the labor of studying, selecting and preparing for the press everything of interest that still remained in the great mass of manuscript that Gauss left behind. Professor Klein, with customary energy and dispatch, has made the necessary arrangements to bring Schering's labor to a prompt termination.

Four new volumes are planned. Professor Brendel, of Göttingen, is editor-in-chief, having the editorial supervision of the entire undertaking. He has also, in particular, the preparation of volume VII. This is to contain the final edition of the Theoria motus and Gauss's voluminous work on the theory of perturbation of the smaller planets, the theory of the moon, etc.

Volume VIII is the volume under review and will be spoken of later.

The contents of volume IX fall into two parts. One of these is to be devoted to mathematical physics, and is in charge of Professor Wiechert, director of the magnetic observatory in

Göttingen. The other part in charge of Professors Börsch and Krüger of the geodetic Centralinstitut in Potsdam, is to contain matter relating to geodesy.

Volume X will contain extracts from Gauss's very extensive scientific correspondence, also biographical notes. Finally a supplementary volume is planned to give a complete index of the preceding ten volumes.

In this connection it is interesting to learn that the movement inaugurated by Professor Klein to collect letters written by or to Gauss, copies of his lectures, scattered manuscripts, note books, in short anything and everything which will help posterity to understand and appreciate Gauss's genius, has been eminently successful. The so-called Gauss rooms in the observatory at Göttingen now serve to house Gauss's papers, and memorabilia relating to him. These, when finally arranged, will without doubt be made accessible to the public and form a fresh inducement, if indeed any were needed, for the student of mathematics to visit Göttingen.

We turn now to the consideration of volume VIII, the first of the new series to appear. Its contents may be regarded as addenda to the first three volumes of Schering's edition, and to the fourth, excluding geodesy.

The new material falls under the following heads: Arithmetic and algebra, pp. 3–32; Analysis and function theory, pp. 36–117; all under the charge of Professor Fricke, of Braunschweig. Numerical calculation, pp. 121–130; Calculus of probabilities, pp. 133–156, in care of Professors Börsch and Krüger, of Potsdam. Foundations of geometry, pp. 159–268; Geometria situs, pp. 271–286; Problems and theorems in elementary geometry, pp. 289–300; Application of complex numbers to geometry, pp. 303–362; Theory of surfaces, pp. 365–449, all under the care of Professor Stäckel, in Kiel, with the exception of a single paper on developable surfaces, three pages in length, by Professor Weingarten, of Charlottenburg.

As the reader may know, the first three volumes of Schering's edition, devoted chiefly to the theory of numbers and algebra, the elliptic and hypergeometric functions, contained a very large amount of material taken from Gauss's posthumous papers. It is not astonishing, then, that a second gleaning does not bring to light much new matter on these topics. Interesting however are the following. First we note two short fragments on cubic and biquadratic residues. In the two

epoch-making memoirs on biquadratic residues published in 1828 and 1832 Gauss took the momentous step of introducing imaginary numbers into the theory of numbers. In two places in these memoirs Gauss states that he began to study the higher laws of reciprocity in 1805 and was soon led to introduce the roots of unity into his speculations as the true basis of this theory. The above notes have now an especial interest to the devotees of the sublime science, as Gauss often styled the higher arithmetic, because they belong to this early epoch, 1805–1808. Let us observe in passing that the ideas contained in the memoir of 1832, properly generalized by Kummer, Dedekind and Kronecker, have led to the modern theory of algebraic numbers.

In the section devoted to Analysis and theory of functions, three papers attract our attention. One is the celebrated letter of 1811 to Bessel. Here Gauss discusses summarily the nature of a function and an integral taken between complex limits. In the first place, he advocates warmly the introduction of the complex variable into analysis. He urges:* "Es ist hier nicht von praktischem Nutzen die Rede, sondern die Analysis ist mir eine selbständige Wissenschaft die durch Zurücksetzung jener fingirten Grössen ausserordentlich an Schönheit und Rundung verlieren und alle Augenblick Wahrheiten, die sonst allgemein gelten, höchst lästige Beschränkungen beizufügen genötigt sein würde."

After stating Cauchy's theorem that $\int f(x)dx$ around a closed contour containing no singular points is zero he remarks†: "Dies ist ein sehr schöner Lehrsatz dessen eben nicht schweren Beweis ich bei einer schicklichen Gelegenheit geben werde. Er hängt mit schönen andern Wahrheiten, die Entwickelungen in Reihen betreffend, zusammen."

The latter part of this quotation suggests at once far-reaching consequences: Does Gauss have in mind the developments of Taylor, Laurent and Fourier? In this letter Gauss also emphasizes the importance of those functions which we call to-day one valued integral transcendental functions. In place of the function Soldner had introduced, viz., $\int(\log x)^{-1}dx$, he suggests $\int(e^x - 1)x^{-1}dx$.

* P. 90. We take this occasion to remark that all references without indication of the title refer to Gauss's Works, Vol. VIII.

† P. 91.

The second paper, which Fricke dates about 1800, relates to the inversion of the elliptic integral

$$u = \int \frac{dx}{\sqrt{(1-x^2)(1-\mu^2x^2)}}.$$

We find to our surprise that the development given by Gauss is nothing more nor less than Weierstrass's celebrated expression of x as the quotient of two integral transcendental functions, viz., $Al_1(u)$, $Al_0(u)$.

The third interesting feature in the section are three fragments on elliptic modular functions. If the mathematical world was filled with astonishment in 1827–28 when it became gradually known that Gauss had had in his possession for thirty years a good part of Abel's and Jacobi's results, we to-day are hardly less astonished to learn that Gauss certainly knew some of the modern theory of the elliptic modular functions. In fact, on page 104 we find for example a modular figure with the angles $\pi/4$, $\pi/4$, $\pi/4$, complete with its *orthogonal circle*. It is worthy of note that on page 478 of volume III appears another figure very familiar to students of the modular functions, viz., the fundamental domain of the modulus $-ik^2$. But as this volume was published in 1876, while Fuchs's and Dedekind's papers, which gave birth to the modular functions, did not appear till the following year, the figure last mentioned could not mean much to Schering, otherwise he would have searched the manuscripts of Gauss more carefully for other fragments of this nature.

The section devoted to the calculus of probabilities contains some interesting letters of Gauss. The theory of least squares, so important in the adjustment of observations, was discovered independently by Legendre and Gauss. The first published account of it is found in Legendre's Nouvelles méthodes pour la détermination des orbites des comètes, which appeared in 1806. In the Theoria motus, published in 1809, Gauss claimed to have discovered it in 1795. A controversy arose, very painful to the friends of Gauss. For example, Legendre in a letter* to Jacobi dated 1827, speaking of Gauss's claims of priority in the elliptic functions, breaks out: "Comment se fait-il que M. Gauss ait osé vous faire dire que la plupart de vos théorèmes lui etait connus, et qu'il en avait fait la decouverte dès 1808? * * * Cet

*Jacobi, Werke, Vol. I, p. 398.

excès d'impudence n'est pas croyable de la part d'un homme qui a assez de mérite personnel pour n'avoir pas besoin de s'approprier les découvertes des autres * * * Mais c'est le même homme qui en 1801 voulut s'attribuer la découverte de la loi de reciprocité publiée en 1785 et qui voulut s'emparer en 1809 de la méthode des moindre carrés publiée en 1805. D'autres exemples se trouveraient en d'autres lieux, mais un homme d'honneur doit se garder de les imiter."

Such direct accusations were of course not published, but they were circulated in private. Enough was said in print and in private to cause Gauss's friend Schumacher to write him (1831) as follows:* "Da Sie die Resultate Ihrer Rechnung geben, so scheint es mir, ist es leicht zu zeigen dass diese durch die Methode der kleinsten Quadrate abgeleitet sind. Zach lebt zudem noch, und hat gewiss Ihren Brief aufgehoben. Finden Sie es nicht der Mühe werth endlich die Sache einmal, selbst gegen die mir vor allen widerlichen *höflichen* Zweifel der Franzosen unwidersprechlich abzumachen"? To this Gauss replies:† "Ich glaube Ihnen schon einmal geschrieben zu haben, dass ich auf keinen Fall diese Stelle worin die Methode zum erstenmale öffentlich angedeutet ist releviren werde, auch nicht wünsche, dass einer meiner Freunde mit meiner Zustimmung es thue. Diess hiesse *anerkennen als bedürfe* meine Anzeige (Theoria Motus Corporum Coelestium) dass ich seit 1794 diese Methode vielfach gebraucht habe, einer Rechtfertigung, und dazu werde ich mich nie verstehen. Als Olbers attestirte dass ich ihm 1803 die ganze Methode mitgetheilt habe, war diess zwar gut gemeint: hätte er mich aber vorher gefragt, so würde ich es hautement gemissbilligt haben."

The grandeur and inflexibility of Gauss's nature is finely illustrated by this passage. In a later letter‡ (1840) to Schumacher we find Gauss's own estimate of the importance of the discovery of least squares. He writes: "Sie wissen dass ich selbst auf das von mir seit 1794 gebrauchte Verfahren * * * niemals grossen Werth gelegt habe. Verstehen Sie mich recht; nicht in Beziehung auf den grossen Nutzen den sie leistet, der ist klar genug, aber *danach* taxire *ich* die Dinge nicht. Sondern deshalb oder in so fern legte ich nicht viel Werth darauf, als vom ersten Anfang an der Gedanke mir so

* P. 137.
† P. 138.
‡ P. 141.

natürlich, so äusserst nahe liegend schien dass ich nicht im Geringsten zweifelte, viele Personen, die mit Zahlenrechnung zu verkehren gehabt, müssten von selbst auf einen solchen Kunstgriff gekommen sein und ihn gebraucht haben ohne deswegen es der Mühe werth zu halten viel Aufhebens von einer so natürlichen Sache zu machen."

The precocious genius of Gauss will stand forth when we recall that he was but seventeen years old when he discovered his method of least squares.

Before passing on to the next section we must call attention to a curious mishap which has befallen the editors. Thirty of the hundred pages given to the section of analysis and theory of functions are occupied by a memoir entitled: "De integratione formulae differentialis $(1 + n \cos \phi)^r d\phi$." The manuscript of this memoir was discovered in 1893 and published by Schering with explanatory notes in the *Göttinger Nachrichten*. Fricke in a note * remarks: "Es fehlt in der Urschrift der Name des Verfassers und eben so auch eine Angabe über Ort, und Zeit der Abfassung. In dessen beseitigte der Vergleich sonstiger Manuscripte von Gauss mit der vorliegenden Handschrift jeden Zweifel welcher sonst vielleicht an Gauss' Autorschaft bestehen könnte." Fricke places it in the year 1795 or before, *i. e.*, before or at the beginning of Gauss's student days in Göttingen. It turns out now, however, that the memoir was not written by Gauss at all, but by a certain Thibaut, who presented it to the Academy of Göttingen in the year 1799.†

We come now to the most interesting section of the book, the Foundations of Geometry, or the Non-Euclidean Geometry. Here as in the case of the elliptic functions Gauss had explored far into a territory a quarter of a century or more in advance of his contemporaries and for years was in possession of epoch-making results. But he never published anything on this subject. In letters to a few of his intimate friends he would occasionally explain some of his views or indicate in a fragmentary manner some of his results. Also on several occasions he reviewed in the *Göttinger Anzeigen* works touching on the theory of parallels. All that is known of Gauss's views relating to the non-euclidean geometry has been collected and put in this section.

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*P. 64.

† F. Klein *Göttinger Nachrichten* (*Geschäftliche Mitth.*), 1902, p. 12.

Gauss's reason for not publishing his results will be made clear from the following extracts. In a letter* to his student friend, W. Bolyai (1799), he writes: "Mach' doch ja Deine Arbeit bald bekannt; gewiss wirst Du dafür den Dank nicht zwar des grossen Publicums (worunter auch mancher gehört, der für einen geschickten Mathematiker gehalten wird) einernten, denn ich überzeuge mich immer mehr dass die Zahl wahrer Geometer äusserst gering ist, und die meisten die Schwierigkeiten bei solchen Arbeiten weder beurtheilen noch selbst einmal sie verstehen können — aber gewiss den Dank aller derer, deren Urtheil Dir allein wirklich schätzbar sein kann."

In 1818 Gerling writes† Gauss that he intends to prepare a new edition of Lorenz's Reine Mathematik and wishes to know Gauss's opinion "wie es mit der Parallelentheorie wohl am besten zu halten ist." Gauss replies:‡ "Ich freue mich, dass Sie den Muth haben sich so auszudrücken, als wenn Sie die Möglichkeit, dass unsere Parallelentheorie, mithin unsere ganze Geometrie, falsch wäre, anerkennten. Aber die Wespen, deren Nest Sie aufstören werden Ihnen um den Kopf fliegen."

In a letter to Bessel (1829) Gauss writes: § "Auch hier habe ich manches noch weiter consolidirt und meine Überzeugung dass wir die Geometrie nicht vollständig a priori begründen können, ist, wo möglich, noch fester geworden. Inzwischen werde ich wohl noch lange nicht dazu kommen meine *sehr ausgedehnten* Untersuchungen darüber zur öffentlichen Bekanntmachung auszuarbeiten, und vielleicht wird dies auch bei meinen Lebzeiten nie geschehen, da ich das Geschrei der Böotier scheue, wenn ich meine Ansicht *ganz* aussprechen wollte."

The wasps and the Bœotians that Gauss here refers to are probably not so much the mathematicians as the philosophers. As will be recalled, Kant in his Kritik der reinen Vernunft (1781) had endeavored to show that our notions of space were *essentially* à priori, *i. e.*, independent of all experience. In fact his demonstration of the possibility of synthetic judgments à priori, one of the corner stones of his philosophy,‖ depends largely on this view of space.

* P. 159.
† P. 178.
‡ P. 179.
§ P. 200.
‖ "Auf der Auflösung dieser Aufgabe oder einem genugthuenden Beweise, dass die Möglichkeit die sie erklärt zu wissen verlangt in der That gar nicht stattfinden, beruht nun das Stehen und Fallen der Metaphysik." Kritik der reinen Vernunft, ed. B. Erdmann (1884), p. 41.

Gauss's opinion regarding the à priori character of our notions of space is perhaps seen best from the following extract of a letter to Bessel (1830). He writes:* "Wahre Freude hat mir die Leichtigkeit gemacht, mit der Sie in meine Ansichten über die Geometrie eingegangen sind, zumal da so wenige offenen Sinn dafür haben. Nach meiner innigsten Überzeugung hat die Raumlehre in unserem Wissen a priori eine ganz andere Stellung wie die reine Grössenlehre; es geht unserer Kenntniss von jener durchaus diejenige vollständige Überzeugung von ihrer Notwendigkeit (also auch von ihrer absoluten Wahrheit) ab die der letzern eigen ist; wir müssen in Demuth zugeben dass, wenn die Zahl *bloss* unsers Geistes Product ist, der Raum auch ausser unserm Geiste eine Realität hat der wir à priori ihre Gesetze nicht vollständig vorschreiben können."

His scorn for the philosophers is manifested in a letter to Taurinus (1824); he writes,† speaking of some of the peculiarities of the non-euclidean geometry: "Aber mir deucht, wir wissen, trotz der nichtssagenden Wort-Weisheit der Metaphysiker eigentlich zu wenig oder gar nichts über das wahre Wesen des Raumes, als dass wir etwas uns unnatürlich vorkommendes mit *Absolut Unmöglich* verwechseln dürfen."

Again in a letter to Schumacher (1846), speaking of the notion of right and left, he writes:‡ "Welche Geltung diese Sache in der Metaphysik hat, und dass ich darin die schlagende Widerlegung von Kants Einbildung finde, der Raum sei *bloss* die Form unserer äussern Anschauung habe ich succinct in den Göttingischen Gelehrten Anzeigen 1831, S. 635 ausgesprochen."

We have expressed our astonishment at Gauss's reluctance to prepare for publication the results of his speculations. We now wish to cite a couple of passages which exhibit another trait of his character, the utter lack of envy he displays when his results have been discovered by others and the carelessness he constantly manifests in establishing his rights of priority.

In a letter (1804) to W. Bolyai,§ speaking of his and Bolyai's investigations in non-euclidean geometry, he writes: "Ich habe zwar noch immer die Hoffnung dass jene Klippen einst und noch vor meinem Ende, eine Durchfart erlauben werden. Indess habe ich jetzt so manche andere Beschäftigungen vor

* P. 201.
† P. 187.
‡ P. 247.
§ P. 160.

der Hand dass ich gegenwärtig daran nicht denken kann, und glaube mir es soll mich herzlich freuen wenn Du mir zuvorkommst, und es Dir gelingt alle Hindernisse zu übersteigen. Ich würde dann mit der innigsten Freude alles thun, um Dein Verdienst gelten zu machen und ins Licht zu stellen so viel in meinen Kräften steht."

Again on the same subject in a letter to Gerling (1819) he writes:* "Die Notiz von Herrn Professor Schweikart hat mir ungemein viel Vergnügen gemacht und ich bitte ihm darüber von mir recht viel Schönes zu sagen. Es ist mir fast alles aus der Seele geschrieben."

Finally, the following extract† from a letter to W. Bolyai (1830): "Jetzt Einiges über die Arbeit Deines Sohnes. Wenn ich damit anfange '*dass ich solche nicht loben darf*': so wirst Du wohl einen Augenblick stutzen: aber ich kann nicht anders; sie loben hiesse mich selbst loben: denn der ganze Inhalt, der Weg, den Dein Sohn eingeschlagen hat, und die Resultate, zu denen er geführt ist, kommen fast durchgehends mit meinen eigenen, zum Theile seit 30–35 Jahren angestellten Meditationen überein. In der That bin ich dadurch auf das Aeusserste überrascht. Mein Vorsatz war von meiner eigenen Arbeit, von der übrigens bis jetzt wenig zu Papier gebracht war, bei meinen Lebzeiten gar nichts bekannt werden zu lassen. Die meisten Menschen haben gar nicht den rechten Sinn für das, worauf es dabei ankommt, und ich habe nur wenige Menschen gefunden, die das, was ich ihnen mittheilte, mit besonderm Interesse aufnahmen. Um das zu können, muss man erst recht lebendig gefühlt haben, was eigentlich fehlt, und darüber sind die meisten Menschen ganz unklar. Dagegen war meine Absicht, mit der Zeit alles so zu Papier zu bringen, dass es wenigstens mit mir dereinst nicht unterginge. Sehr bin ich also überrascht, dass diese Bemühung mir nun erspart werden kann, und höchst erfreulich ist es mir dass gerade der Sohn meines alten Freundes es ist der mir auf eine so merkwürdige Art zuvorgekommen ist."

Before leaving this section we must mention two fragments which give rise to bold surmises. We recall that the advance that Riemann made in his Habilitationsschrift (1854) "Über die Hypothesen welche der Geometrie zu Grunde liegen" ‡ was

* P. 181.
† P. 220.
‡ Riemann's Werke, p. 254 seq.

due in part to his introduction of differential geometry into his investigations, whereas the Bolyais, Lobachevsky and Gauss (certainly in his early work) employed the synthetic methods of Euclid.

Now in one of the above mentioned fragments Gauss treats this problem: Assuming that the geometry of Euclid is valid for infinitely small triangles, determine the relations between the sides and the angles of finite triangles. This leads to a differential equation whose constant of integration Gauss represents by k, which is precisely the letter he employs in his Disquisitiones generales circa superficies curvas to represent the measure of curvature. In the second fragment we have a deduction of the equations of the pseudosphere which Gauss calls "das Gegenstück der Kugel." There is in these two fragments no direct evidence that Gauss saw that his non-euclidean geometry found in the pseudosphere a geometric substratum, nor that he saw the intimate relation between the non-euclidean geometry and his theory of developable surfaces; but the supposition lies very near and is not improbable.

We pass over the intermediate sections in order to employ the remaining space at our disposal to touch on a topic of great interest to the English-speaking race, viz., Gauss's relation to linear associative algebra in general and to quaternions in particular.

We are all familiar with Hamilton's efforts, extending over a period of fifteen years, to discover a system of numbers in three units which would bear the same relation to space as the ordinary complex numbers $x + iy$ do to the plane, and which at the same time obey the formal laws of ordinary algebra. Finally in 1843 he discovered the quaternions. Meantime, Gauss had considered the same problem. In his report in the *Göttinger Anzeigen* (1831) of his memoirs on biquadratic residues which we have already referred to, Gauss after discussing the nature of the complex numbers $x + iy$ in a masterly manner of closes his remarks with the following oracular statemen:* "Wir haben geglaubt, den Freunden der Mathematik durch diese kurze Darstellung der Hauptmomente einer neuen Theorie der sogenannten imaginären Grössen einen Dienst zu erweisen. Hat man diesen Gegenstand bisher aus einem falschen Gesichtspunkt betrachtet, und eine geheimnissvolle Dunkelheit dabei gefunden, so ist diess grössentheils den wenig

* Volume II, pp. 177.

schicklichen Benennungen zuzuschreiben. * * * Der Verfasser hat sich vorbehalten den Gegenstand welcher in der vorliegenden Abhandlung eigentlich nur gelegentlich berührt ist, künftig vollständiger zu bearbeiten, wo dann auch die Frage warum die Relationen zwischen Dingen die eine Mannigfaltigkeit von mehr als zwei Dimensionen darbieten, nicht noch andere in der allgemein Arithmetik zulässige Arten von Grössen liefern können, ihre Beantwortung finden wird."

From this passage we see that Gauss already in 1831 had recognized the non-existence of a number system such as Hamilton so long vainly sought to find. The question at once suggests itself: If Gauss was able to state a theorem of such fundamental importance in the theory of higher complex number systems, how far did his researches reach, what was their nature? The new volume of the Werke, while it still leaves us entirely in the dark on this subject, brings however one great surprise. In a fragment* entitled Transformationen des Raumes we find the multiplication table of quaternions. This note is dated by Stäckel, 1819. That the multiplication is not commutative is noted page 360 and the relation between quaternions and spherical triangles is noted pages 360–361. To prevent a possible disappointment to any one referring to these passages we observe that Gauss does not employ the usual quaternion notation $a + bi + cj + dk$ but the earlier notation of Hamilton (a, b, c, d). The introduction of certain symbols for the units (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1) is entirely unnecessary and is made merely for convenience. From a theoretical standpoint the subject even wins in clearness by leaving them out at the start.

Apropos of quaternions we cannot resist the temptation to cite a passage from a letter† to Schumacher (1843) relating to Möbius's Barycentric Calculus. What Gauss here says of the barycentric calculus can be applied directly to quaternions and the vector analysis, a subject dear to many English and American mathematicians. After mentioning the prejudice he first had toward Möbius's work and his doubt "ob es der Mühe werth sei, eine recht artig ausgesonnene Rechnungsweise sich anzueignen, wenn man durch dieselbe nichts leisten könne was sich nicht eben so leicht ohne sie leisten lasse," follows this remarkable passage which Hamilton and Gibbs would

*P. 358.
†P. 297.

doubtless subscribe to most cordially: "Ueberhaupt verhält es sich mit allen solchen neuen Calculs so, dass man durch sie nichts leisten kann, was nicht ohne sie zu leisten wäre; der Vortheil ist aber der, dass, wenn ein solcher Calcul dem innersten Wesen vielfach vorkommender Bedürfnisse correspondirt, jeder, der sich ihn ganz angeeignet hat, auch ohne die gleichsam unbewussten Inspirationen des Genies die niemand erzwingen kann, die dahin gehörigen Aufgaben lösen, ja selbst in so verwickelten Fällen gleichsam mechanisch lösen kann, wo ohne eine solche Hülfe auch das Genie ohnmächtig wird. * * * Es werden durch solche Conceptionen unzählige Aufgaben, die sonst vereinzelt stehen, und jedesmal neue Efforts (kleinere oder grössere) des Erfindungsgeistes erfordern, gleichsam zu einem organischen Reiche."

In closing, we wish to praise the care the editors have taken in preparing the contents of the present volume for publication. As far as possible the dates of the various papers have been ascertained. Obscure passages have been explained and errors either corrected or pointed out. There are instances where the editors have been too sparing with their commentary, but they have doubtless preferred to err on the side of brevity rather than diffuseness.

Only one oversight has been noticed by the reviewer. In the extract printed on page 247, Gauss makes reference to his "schlagende Widerlegung" of Kant's theory of space which appeared in the *Göttinger Anzeigen* for 1831. This is certainly a matter of some importance and a reference should have been given where readers who have not the files of the *Anzeigen* at hand — and their number is relatively small — can find it in the Werke. The passage Gauss has in mind is to be found in volume 2, page 177 and footnote of the Werke. In this connection we wish to express the hope that further volumes will give *exact* references to the places where each particular paper was first published. This is only partially done in the volumes which have so far appeared. Indeed it seems desirable that this deficiency in the early volumes be made good in the supplementary volume.

In regard to misprints, we have observed but one, viz., page 160, last line: omit one *herzlich*.

To sum up, we may say that the volume under review takes its place worthily besides the six preceding volumes which

enjoyed the loving attention of one who sacrificed a large part of his scientific activity in erecting a lasting monument to the memory of Germany's princeps mathematicorum.

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ANALYTIC PROJECTIVE GEOMETRY.

Lehrbuch der analytischen Geometrie in homogenen Koordinaten. Von WILHELM KILLING. 2 Teile, 8vo. Paderborn, F. Schöningh. Teil 1: *Die ebene Geometrie*, 1900. xiii + 220 pp. Teil 2: *Die Geometrie des Raumes*, 1901. viii + 361 pp.

THE first and perhaps the most noticeable feature of Professor Killing's text-book on the analytic geometry of homogeneous coördinates is the elaborately methodical arrangement. The line of march has been laid out with extreme care. Notice, for example, the titles and page numbers of the first few sections of the two volumes:

In the plane, Vol. 1.	In space, Vol. 2.
§ 1. Theory of cross ratio, p. 1.	§ 1. Division of dihedral angles, p. 1.
§ 2. The coördinate triangle, p. 8.	§ 2. The coördinate tetrahedron, p. 7.
§ 3. The straight line, p. 13.	§ 3. The straight line and plane, p. 13.
§ 4. The perpendiculars dropped upon a straight line from the vertices of the coördinate triangle, p. 19.	§ 4. The perpendiculars dropped upon a plane from the vertices of the coördinate tetrahedron, p. 21.
§ 5. The most general trimetric coördinates, p. 25.	§ 5. The most general tetrahedral coördinates, p. 29.
§ 6. The ratios of the coördinates as cross ratios, p. 37.	§ 6. The elements at infinity, p. 38.
§ 8. The elements at infinity, p. 48.	§ 7. The quotients of the coördinates as cross ratios, p. 44.
Total number of pages, 49.	Total number of pages, 52.

There is no necessity of stopping the parallel column here. It might run, with very few exceptions, to the end of the two volumes. Each closes with the treatment of confocal quadrics; the treatment in space is, of course, more elaborate than that required in the plane, but otherwise exactly parallel to it. The differences between the texts of Volume 1 and Volume 2 in the first fifty pages are of about the same order as the difference in the sectional headings quoted above. The changes are of two kinds: Those that are necessitated by the subject — the substitution of "line and plane" for "line," of four variables or four equations for three; and those changes which seem unnecessary — the substitution of the word "quotient" for "ratio" in the sectional headings 6 and 7, and the rearrangement of the order of the sections where it is of no particular importance.

This is carrying method to a painful extreme. There can be little excuse for spending 52 pages of Volume 2 upon those sections which correspond exactly to the 49 pages of Volume 1. The author might, at least, have economized in Volume 2 enough to save the three extra pages needed by the constantly recurring fourth variable and fourth equation. One could throw away the first part and commence on the second without experiencing any particular difficulty. So much elaboration of method is quite unnecessary. The book is well named a *Lehrbuch*, and one can almost hear, almost see the *Lehrmeister* as he reaches the second part say to his pupils: "I have explained the first volume with all care. Here is the second. Read it. It is like the first volume. Toward the end there are a few new things — no, not new, only a little less familiar, a little harder to generalize. If you find difficulty, wake me." And it would be a worthless student who need wake his master. More likely he would himself fall asleep over the monotony, the lack of opportunities to employ his wits and imagination.

Apart from this feature and from its antiquated typography, with spaced letters instead of italics and formulæ in the same font as the text, the book is good. The general plan is excellent; the scope has been chosen with judgment. The subjects treated are not too many nor too few. The methods of analysis are for the most part neat. The author has exercised great care in treating the matter of sign, which is so confusing to the beginner in connection with the coördinate triangle. Although the book is confessedly analytic there is often a glimmer, unfortu-

nately only a glimmer, of truly geometric reasoning. The large numbers of exercises distributed through the text, some at the end of each section, have been selected with especial care, assorted, and graded. For this reason the book must be more than usually helpful to a teacher, especially a young and inexperienced teacher, in preparing a course on projective geometry. To a student, who works without a teacher, who wishes to supplement a meager knowledge or even to gain from the beginning some idea of projective geometry, this book with its careful explanations and its problems will prove a great boon. It offers but little difficulty. It is so easy, elementary and slow; but withal, when one has finished it, so inclusive.

A comparison of the book under review with Duporcq's Premiers principes de géométrie moderne * is very instructive and interesting. Both treat projective geometry, though Duporcq concludes with a chapter on other transformations. But the methods are entirely different. The one is methodical, the other not at all; the one painstakingly accurate in demonstration, the other often hasty and fond of relying upon the sweep of intuition; the one analytic, the other geometric with only a background of analysis, like Chasles; the one assuring, the other stimulating. The fact is that a student of Killing would know that he had at his command a perfectly definite method. On meeting a problem he could work out a solution and probably put it in good form. Yet he very likely would not be one whit a geometer, but merely an analyst. The student of Duporcq could hardly get along without a teacher to help him. He would be quite unable to say whether he had a method at his command or not. He would know many theorems, and if he came to a problem he might or might not solve it. If he solved it he would not know why he had succeeded, nor in what section of geometry to place it. Yet he would think geometrically. The two books are complements of each other. The student of either would find in the other just what he lacked. If he were studying alone, he had best begin with Killing; if with a teacher, with Duporcq.

The place of trilinear and tetrahedral coördinates, the place of projective geometry in general in our mathematical education, has changed considerably since the first appearance of

* Reviewed in the BULLETIN, vol. 6, p. 254.

those classics — Salmon's Conic Sections and Clebsch's Vorlesungen über Geometrie — which, with their various translations into other languages, have done so much to influence the subsequent treatment of projective geometry. Thirty or forty years ago the group of projective transformations, then scarcely recognized as a group, was playing a well-nigh all-important rôle in mathematics owing to its connection with the so-called higher algebra. The vast theory of invariants — invariants of the projective group — was in building. Cayley and Clebsch were doing their great work. The particular enthusiasm of that day has passed by. As one looks over the announcements of the different universities he looks almost in vain for a course upon this theory of invariants. The theory of invariants now means a part of Lie's theory of groups.

The projective transformation itself seems less important. Inversion with its applications to the theory of functions, to potential, to all problems of conformality demands constantly more attention. Moreover there is the vast body of transformations brought to our notice by Lie under the name of contact transformations, of which even a single one, the line-sphere transformation, is of great importance. Thus even if we look only at the field of geometry we must grant that the projective transformation cannot hope to occupy so much of one's time and thoughts as formerly. And when we take into consideration other fields which have been springing up so rapidly and which contain scarcely the slightest reference to projective geometry, we may well wonder whether or not this subject deserves any special consideration, whether we are not giving it too much attention, whether in reality we need it at all except as a simple example of a group of transformations with very interesting properties.

Certain it is that the elaborate volumes on trilinear and tetrahedral coördinates are less attractive now. Room and time must be left for circle and sphere and line (in space) coördinates. Certain it is, too, that now, after the passing of the formal invariant theory of the projective group, projective geometry is taking its place beside other important geometries and we must in every way try to present its essentials as rapidly as possible and without any hope of elaborating its details. The other geometries cannot longer be neglected.

That, however, which will always render projective geometry the best introduction to other geometries is its geometric sim-

plicity and elegance. Its conceptions, its transformations are more easily visualized. The figures with which it deals are of the simplest. The methods of treatment have been so developed that one may proceed analytically or synthetically and may thus acquire at the same time both analytic and synthetic facility. To mix these two methods is distasteful to some persons. But the fact is that many theorems are more easily proved synthetically, many others analytically. In the latter case one must needs commence at the beginning with a coördinate system and work up; in the former one starts at least half way up, resting upon a large basis of known theorems. Each method has its advantages. It is still better to mingle the methods, changing from one to the other whenever the change illuminates the problem or facilitates its solution. In other words, nowadays, when a time and space limit is imposed upon us, we must at first leave aside puristic considerations of all analysis or all synthesis, and must content ourselves with developing both the analytic and synthetic methods in less time than formerly was allotted to each.

For this purpose many properties of trilinear coördinates become useless. It is of little value to know that the trilinear coördinates of a point are proportional to constant multiples of the perpendicular distances of the point from the sides of the triangle of reference. The property is not projective. On the other hand it is important to know that the ratios of the coördinates are cross ratios at the vertices or upon the sides of the triangle of reference. This property is projective and in fact assures us that the coördinate system chosen is the most general.

To shorten further and simplify the analytic presentation of projective geometry it is necessary carefully to isolate the difficulties which appear in the subject from the student's standpoint in passing from cartesian geometry. These difficulties seem to be three: The question of infinity, the fact that homogeneous instead of non-homogeneous coördinates are used, the idea and use of line coördinates in the plane or plane coördinates in space are very real difficulties. Each is a new idea to the student. They occur again and again in other geometries than projective. Beyond these three there is scarcely any other difficulty unless it be the use of abridged notation in avoiding the solution of equations. For in projective geometry as in other higher geometries one constantly aims at using expres-

sions like $u + \lambda v = 0$, $uv = 0$, determinants, etc., as much as possible to avoid tedious analysis. This can be done, although it usually is not done, in ordinary cartesian geometry.

For the past two years it has fallen to my lot to teach plane projective geometry to a class of students who know no other geometry than the euclidean and the elements of the cartesian. The allowance of time has been a half-year, forty-five lessons. This is short — perhaps too short. But when we consider the vast number of subjects, the calculus, mechanics, geometry, algebra, and perhaps physics, which demand the attention of an intermediate student of mathematics, say a junior in college, one is perhaps willing to grant that projective geometry cannot claim more. As many others may encounter similar problems in instruction the account of my experience, which comes in naturally in connection with my criticisms of Killing's Geometry, may be pardoned.

To have followed a book such as Killing's or Salmon-Fiedler's would have been to give the student not more than a bare working knowledge of trilinear coördinates of point and line. To educate his geometric insight to any great extent would have been impossible. To have followed exclusively the synthetic methods would have been to leave the student not very well grounded in them, and absolutely without the basis of analysis. It was evidently necessary, and only fair, to combine the synthetic and analytic methods.

To get the student back to where he was when he finished his Euclid, to get him again to thinking about geometric objects and geometric methods, to relieve him of the idea that geometry beyond Euclid consists merely of sets of algebraic equations of the first and second degree, it was necessary to begin with pure synthetic geometry. The subjects treated were:* Harmonic elements; the principle of duality; the elements at infinity; projective relations upon a line; involution; collineation and correlation in the plane; polarity; the conic; projective relations between ranges or pencils upon a conic. This much insures that the student thinks geometrically, that he has settled his mind about infinity and that he can solve a tolerably large number of problems by means of the fundamental theorems which he has learned.

*See my paper on "The synthetic treatment of conics" in the BULLETIN, 1903, p. 248.

The analytic treatment is then conducted as follows: Abridged notation; the introduction of homogeneous rectangular coördinates $x : y : t$, defined by $X = x : t$, $Y = y : t$; line coördinates defined as the negative reciprocals of the intercepts of a line upon the axes X and Y; homogeneous line coördinates; the cross ratio, and its independence of projection; the fact that the ratios of the homogeneous point or line coördinates are cross ratios upon the axes or at infinity; the projection of the rectangular axes of reference and the line at infinity and the unit point into an arbitrary triangle and arbitrary point; the point and line coördinates which result; practice in the use of these coördinates. It will be seen that the difficulties of abridged notation, homogeneous coördinates and line coördinates were each treated before definitely leaving the familiar system of rectangular cartesian coördinates. It will be seen furthermore that trilinear coördinates, together with their triangle and unit point of reference, were introduced as a mere projection of customary coördinates. The two facts which allow this are the theory of collineations as explained in synthetic geometry and the invariance of the cross ratio. It is seen also that the analysis in trilinear coördinates is identical with the analysis in rectangular homogeneous coördinates, if the triangle of reference and the figure to be investigated be both projected back on the plane of X and Y. This numerical identity of the calculations in the two cases is a great help in giving the student confidence in his analysis. It is very much more valuable than the ordinary method of deducing the coördinates by the method of perpendiculars. It also avoids the necessity of the proof that the linear equation represents a straight line or point.

Generally after this method of presentation there are still a few lectures remaining for the purpose of review, to say a few words about quadratic and Cremona transformations, to take up the circular points and focal properties, or to outline the generalization to space. The student has at any rate a working knowledge of synthetic and analytic methods in the plane.

To any one who wishes to work up a course such as that outlined, the following three books: Killing's Lehrbuch der analytischen Geometrie, for its analysis; Enriques's Geometria proiettiva, for its synthetic methods; and Duporcq's Premiers principes de géométrie moderne, for its intermediary liveliness and its exhibition of the relations between metric and projective geometry, form a library which is quite sufficient and one

may almost say necessary. There may be other books as good; but for this particular purpose these are not easily improved upon.

EDWIN BIDWELL WILSON.

ÉCOLE NORMALE SUPÉRIEURE,
PARIS, FRANCE,
December 5, 1902.

SHORTER NOTICES.

Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance. By M. CURTZE. Zweiter Theil. *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, XIII Heft. Leipzig, Teubner, 1902. 292 pp. 14 marks.

Abhandlungen zur Geschichte der mathematischen Wissenschaften, XIV Heft, Leipzig, Teubner, 1902. 338 pp. 16 marks.

THE first part of Curtze's Urkunden, forming the twelfth volume of the Abhandlungen, has been reviewed in the BULLETIN * so recently that it is quite surprising to find two new volumes of the series already published. Indeed no better evidence of the present revival of interest in the history of mathematics can be found than is seen in the encouragement recently given to this series founded a quarter of a century ago by Professor M. Cantor. The publication of the first seven volumes extended through a period of nineteen years, while the last seven, including the two under review, have appeared since 1897.

The second part of Herr Curtze's Urkunden is devoted to two interesting manuscripts, one the Practica Geometriæ of Leonardo Mainardi of Cremona, and the other the algebra of Initius Algebras. The first, which also bears the title Leonardi Cremonensis Artis Metrice Practice Compilatio, is a transcript, with German translation, from an Italian codex in the Venetian dialect in the university library at Göttingen. This codex is not unique, for Prince Boncompagni had two Latin manuscripts of the same work; but not only has it never before been published, but Leonardo Mainardi has been practically unknown to historians of mathematics. It consists of fifty folios, of which the first twenty-nine and the last fourteen are here

* Vol. 9, p. 123, Nov., 1902.

transcribed and translated, the others containing merely tables of sines and a *tabula sollis* (sic), that is a table of the lengths of the various days of the year. Of the Boncompagni codices, the older belongs to the fifteenth century, although ascribed by Narducci to the fourteenth — a century before Leonardo was born. The Göttingen codex, which Herr Curtze has carefully compared with the other two, was written in 1488.

Of the life of Leonardo nothing is known save what appears in the Cremona Literata of Franciscus Arisius, published in 1702, who seems to have had access to the later of the Boncompagni codices, and who speaks of "Leonardus Maynardus Insignis Astronomus, Physicus et Mathematicus," as having flourished in LXXXVIII, that is in 1488.

The treatise itself is not on pure geometry, but is devoted to the mensuration of heights, surfaces and solids, with such trigonometry as was necessary for the work in hand. It is divided into three "tractati," the first being introduced with an apology for entering a field already well covered, but with the hope that this "picholo volumen" may prove a valuable compendium. This tractate is devoted to the measurement of heights, depths and lengths; the second discusses polygons and the circle, with their superficial measurement; and the third relates to the volumes of polyhedra and the three round bodies. Of the quaint mathematical terms, which of course include *umbra recta* and *umbra versa*, one of the most interesting is *columna uniforme* for a right circular cylinder or prism, a name with much to commend it. The common rules are, of course, substantially our present formulas, except for such interesting variations now and then as that for the volume of the sphere, $\frac{1}{4}d^2\pi \cdot \frac{2}{3}d$. In general the value of π is taken as $3\frac{1}{7}$, but in one case he gives the interesting approximation $3\frac{3983}{28224}$. This he obtains from the Archimedes limits by this process: $3\frac{10}{71} + \frac{1}{3}(3\frac{1}{7} - 3\frac{10}{71})$, and adds, "e questa e la proporcion del diametro a la circonferentia," interesting as giving the older use of "proportion," as a close approximation, as indicating that Leonardo thought this the exact value, and as inverting the terms of the ratio.

The second part of Herr Curtze's work is entitled "Die Algebra des Initius Algebras ad Ylem geometram magistrum suum." Of this work, almost unknown * to historians, four

*See Cantor, *Vorlesungen über Geschichte der Math.*, vol. 2, p. 589.

manuscripts are extant, no one of them the original. Of these the one in the university library at Göttingen, a manuscript of two hundred and five folios, has been followed by Herr Curtze, but it has been carefully compared with the others (all in the Dresden library) and any material differences have been noted. The book is exceedingly interesting on several accounts, mathematically and historically. Written in the south German dialect in the sixteenth century, it displays considerable knowledge together with a confusion of historic facts, and shows no little acquaintance with the algebra that was just beginning to be revived in Italy. The prologue opens: "Hic hebet sich an das Buch Algebræ,* des grossen Arismetristens, geschrieben zu den zeithen Alexandri vnd Nectanebi, des grossen Grecken vnnd Nigromantis, geschrieben zu Ylem, dem grossen Geometer jn Egypten, jn Arabischer Sprach genant *Gebra vnnd Almuchabola*, das dann bey vns wirdt genant *das Buch von dem Dinge der vnwissenden zall*. Vnd ist aus Arabischer Sprach jn kriechisch transferirt von Archimede, vnnd aus kriechisch jn das Latein von Apuleio, vnd wird genandt bey den Welschen *das Buch de la cosa*, das dann aber wird gesprochen *das Buch von dem ding* * * * Vnnd aus disem Buch finden wir, das der Machomet in seinem Alkoran vermeldet von disen Regeln, vnnd nennet sie auch Gebram vnd Almuchabolam. Sie werden auch gebraucht von den Indiern * * * Sie gaben auch das gemelte Buch durch etliche grunde geleutert, als es Algebras gesagt hatt, vnnd ist geschrieben erstlichen von Algebras zu Ylem, dem grossen Geometer, der do was preceptor oder vorfarn Euclidis des fursten zu Megarien."

This passage is one of the most interesting, for its length, in the history of mathematics. The unknown Initius Algebras appears as one who lived in the fourth century B. C., contemporary with the equally unknown Ylem. The title of Alkhowarizmi's book is given, but the Arabic word *algebra*, which there first appears, is assigned to a period over a thousand years earlier and to the Greek language. Mohammed the prophet is confused with the son of Moses, ben Musa Alkhowarizmi, and the not uncommon mistake is made of taking Euclid of Megara for Euclid the Alexandrian. The early German use of "Welsch," which appears in the "Welsch practice" of the sixteenth century arithmetics, is seen; the

* *I. e.*, the author, Initius Algebras.

uncertainty of name for the comparatively new science is apparent; the "unknown quantity" takes its place in algebra; and India is recognized as a mathematical center. Interesting as is the passage, it is no more so than many others in the work, although no other condenses so much in an equal space.

The work covers the ground of the Arab algebra, with the usual German symbolism of the sixteenth century. But the most noteworthy feature of the entire text is the reference to the solution of the cubic. That the author thought he had a solution is apparent from two of his statements,* in the second of which he expressly says that he will later show how to solve equations of the (modern) forms,

$$a + bx^2 + cx^3 = dx, \quad cx^3 + a = dx + bx^2, \quad cx^3 + dx = bx^2 + a.$$

To be sure this promise is not fulfilled, but for such a statement to be definitely made, by a writer who knew a great deal of algebra, and made before the Ars magna of Cardan appeared, is at least very interesting. It suggests a possibility that, after all, the Tartaglia-Cardan formula was only a rediscovery of some cherished secret of a forgotten Arab writer whose work appears in unrecognizable form in this German manuscript. Such an interpretation, at least, seems to be in Herr Curtze's mind, although to most readers it will doubtless seem more probable that the wish of the writer was father to the thought, and that it was only a dream like that of the alchemist or the circle squarer, that found place in the mind of the algebraist.

The fourteenth volume of the *Abhandlungen* contains three articles. Of these the first is by A. A. Björnbo, entitled "Studien über Melenaos' Sphärik; Beiträge zur Geschichte der Sphärik und Trigonometrie der Griechen," and served in part as the author's inaugural dissertation in 1901. It forms without question the most complete study of Menelaus that has yet been made, not only from the sources already published, but from five of the best-known Latin manuscripts of the Sphere, no Greek copy being extant, and from Arab sources as well. The author has also collated the references to Menelaus by the various Greek and Arab writers, and has critically compared the various printed editions of the work in hand, all with a scholarship that promises well for his future work in the history of mathematics.

* Pages 490, 540.

The well-known relation of the Sphere of Menelaus to the plane geometry of Euclid is brought out with a new and interesting clearness. The author concludes that the concept "spherical triangle" was introduced into mathematics by Menelaus, for although other Greek writers, including Autolycus, Euclid, Hypsicles, and Theodosius of Tripoli, had written upon the sphere, no trace of any trigonometric treatment is found in their writings. Indeed the theory of plane trigonometry may also be said to take its scientific origin in this same work, although of course the science was known before this time, notably by Hipparchus. But interesting as are all of these facts, far more so is the proof adduced to show that Menelaus recognized the projectivity of the anharmonic ratio on the sphere, and stated it as a fact known to his predecessors, probably including Hipparchus. Then with his characteristic thoroughness the author traces this proof through the Arab translators and back through the late Latin, to show how the fact happened to be lost for so many centuries.

The second part of the volume is devoted to certain "Nachträge und Berichtigungen zu 'Die Mathematiker und Astronomen der Araber und ihre Werke'" which appeared in volume 10 of the *Abhandlungen*.

The third part consists of an essay by Karl Bopp, "Antoine Arnauld, der grosse Arnauld, als Mathematiker." The idea has never been entirely wanting that Arnauld was a mathematician of power. Cantor mentions him, although only in relation to a controversy between Leibnitz and Guido Grandi; but even by his own countrymen his contributions to mathematics have long since been forgotten. Arnauld (b. 1612) lived in the time of Louis XIV, and received the name of "le grand Arnauld" because of his philosophical and theological attainments. Indeed it is in his philosophical writings that many of his mathematical ideas are suggested. It is, however, in his *Nouveaux Elémens de Géométrie* (1667) that he appears as a mathematician purely, and it is to this work that Herr Bopp devotes his chief attention. In Besoigne's *Histoire de l'abbaye de Port Royal* (1752) an anecdote of M. Nicole is given to the effect that Pascal having one day showed Arnauld a manuscript of his own on Euclid, the latter objected because the faulty arrangement of the Greek was unchanged. Thereupon Pascal challenged Arnauld to write a better work, a suggestion to which the latter acquiesced, and the "New Elements" was the result.

It is not worth while to enter into the general nature of the work. It was elementary but original. One or two of the most interesting features, however, deserve mention. The author uses in one place the expressions $\frac{1}{10}$ d'x, $\frac{1}{10}$ d'y, to represent these fractional parts of x and y, which leads Herr Bopp to suggest that this French symbolism might have suggested to Leibnitz his dx, an idea that is interesting even if far-fetched. He also goes more extensively into the theory of triangular numbers than had any of his predecessors, a fact which shows the influence of the Pascal-Fermat school and the overstepping of the traditional boundaries of geometry. Euclid's parallel postulate is passed with little question, for, he says, "elle a assez de clarté pour s'en contentes et ce seroit perdre de temps inutilement que de se rompre la tête pour la prouver par un long circuit," showing that his vision in this respect was no clearer than that of his contemporaries. He definitely asserts, however, the impossibility of solving the trisection problem by elementary geometry, "c'est à dire en n'y employant que des lignes droites et circulaires." Here, too, is found, five years before Pascal's publication, the latter's method of complete induction, a method which Maurolycus also understood long before either, but which was not generally appreciated. Such phrases as "it is necessary and sufficient" show that his thought was much ahead of the elementary writers of his period, while many of his proofs had a marked influence on the later French school. His generalization of propositions shed a new light on Euclid, and it is probably not too much to say that the French elementary geometry broke away from the Greek traditions largely through the influence of his initiative. The work closes with Arnauld's contributions to the theory of magic squares, a contribution that was original and powerful, and exceeded in value anything of the kind that had been attempted before his time.

DAVID EUGENE SMITH.

Annuaire pour l'an 1903, publié par le Bureau des Longitudes. Paris, Gauthier-Villars.

THERE is, so far as we know, no scientific annual which contains anything like the amount of information which we find in the Annuaire. It appears to be an attempt to satisfy the needs of everyone except, perhaps, the pure mathematician. It con-

tains all kinds of astronomical information, both of temporary and permanent value for the benefit of the astronomer; information about weights and measures, and different kinds of money for the business man; tables of insurance and population, geography and statistics; all kinds of physical tables with the latest values of the physical constants; and in fact the results for everything that admits of being measured. The progress of science, however, has made the labor of the editors an increasingly difficult one. Each year of publication has seen an addition of perhaps twenty or thirty pages, until the volume has become of such a size that some new plan is necessary in order to keep it within reasonable limits.

This question has been occupying the attention of the editors for the last twelve months. It seemed undesirable to curtail the information to be inserted to any considerable extent, and at the same time it was necessary to avoid further increase. The problem has been solved on the principle of rotation. Much of the information which is given varies little, if at all, from year to year—for example, weights, measures, geography, statistics, etc.; while other portions—for example, tables of the sun and moon—must be given every year. It is thus possible so to divide the information that two successive volumes of the Annuaire shall give everything. In brief, every Annuaire will contain about 330 pages of astronomical information, part of which will be published every year, and the other part every two years. The physical constants will be given in the Annuaires of the even years; geographical and statistical constants in those of the odd years. This arrangement will come into force next year, the present number being complete in itself.

There are no special changes to note. All information is of course brought as far as possible up to date. The appendix contains the following notices: On meteorites and comets by M. Radau—an extended history of the subject up to date; a speech on the connection of science and poetry by M. Janssen; the usual report from the Mont Blanc Observatory by the same; the funeral orations of MM. Bassot and Poincaré on M. A. Cornu; and those of MM. Bassot, Bouquet de la Grye, Loewy, Janssen and Van de Sande Bakhuyzen on M. H. Faye.

ERNEST W. BROWN.

Net Premiums and Reserves on Joint Life Policies, based on the American Table of Mortality graduated by the Makeham Formula, etc., etc. By ARTHUR HUNTER. New York, The Actuarial Society of America, 1902. 4to., 102 pp.

WHILE this book is mainly filled with tables of values relating to life insurance policies based on any combination of two or three lives at risk — and is therefore designed for technical use exclusively — it may interest the readers of the BULLETIN to have pointed out to them somewhat of its scientific basis, and incidentally to note the great expansion of the life insurance business when such works of reference become necessary.

Mortality tables have been developed from crude statistics in many ways — by simple collection of data, by grading these by various summation formulæ, by arbitrary graphic interpolation and the like. Some of the tables thus obtained are incorporated in law, as the "Northampton" for valuing life interests, the "Actuaries" and the "American" for the valuation of life insurance policies. But the only scientific method which gives an algebraic expression for the law of mortality and offers great practical advantages in use has never till now been given legal recognition. This method — the Makeham Formula — gives a curve expressing the law of mortality which is continuous and adheres closely to the original data.

By former methods of graduating the crude data, the resulting tables required to complete the matter in these hundred odd pages would have filled several volumes the size of Webster's Dictionary and have been more than proportionately troublesome to use. This condensation is effected by using a curious property of Makeham's law — that of "uniform seniority" — by which instead of having to use a different table for each different combination of ages of the two lives involved in a compound risk, but one table is needed — with an auxiliary table to show the modified argument by which to enter for each varying combination of lives. In other words, under Makeham's law, for every product of the probabilities of two lives aged x and y each continuing for a year, may be substituted the square of a similar probability for one life z, z being intermediate between x and y.

Makeham's law was first established in 1860, but its practical value is just beginning to be appreciated, in this country at least.

J. M. GAINES.

NOTES.

THE thirteenth regular meeting of the Chicago section of the AMERICAN MATHEMATICAL SOCIETY will be held at Northwestern University, Evanston, Ill., on Saturday, April 11, 1903. Titles, abstracts and time requirements of papers should be in the hands of the Secretary of the Section, Professor T. F. HOLGATE, 617 Hamline St., Evanston, Ill., at least two weeks before the meeting.

THE tenth summer meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at the Massachusetts Institute of Technology, Boston, Mass., during the week beginning August 31. The last four days of the week will be devoted to a colloquium at which courses of lectures will be given as follows:

By Professor E. B. VAN VLECK, six lectures on "Selected topics in the theory of divergent series and continued fractions"; by Professor H. S. WHITE, three lectures, subject to be announced; by Professor F. S. WOODS, three lectures on "The connectivity of non-euclidean space."

A preliminary announcement giving further details concerning the summer meeting and colloquium will be issued in May.

THE Annual Register of the Society, containing a catalogue of the library, has recently been issued. Copies have been mailed to the members of the Society, and extra copies can be obtained from the Secretary.

AT the meeting of the London mathematical society held on February 12, 1903, the following papers were read: By Lieut.-Col. A. J. CUNNINGHAM: "On 4-ic residuality and reciprocity;" by Mr. E. T. DIXON: "Note on a point in the recent paper by Professor D. Hilbert;" by Mr. H. HILTON: "Some properties of binodal quartics;" by Professor A. W. CONWAY: "The field of force due to a moving electron;" by Professor W. BURNSIDE: "An arithmetical theorem connected with the roots of unity and its application to group characteristics."

THE Paris academy of sciences announces the following prizes: The Francœur prize, 1,000 francs, for a discovery or treatise advancing pure or applied mathematics.—The Poncelet prize, 1,000 francs, for that work which in the last ten years

has contributed most to the advance of pure or applied mathematics.—The Grand prize of the mathematical sciences for 1904, 3,000 francs, for perfecting in some important point the study of the convergence of continued algebraic fractions.—The Bordin prize for 1904, 3,000 francs, for developing and perfecting the theory of surfaces applicable to the paraboloid of revolution.—The Vaillant prize for 1904, 4,000 francs, for determining and studying all the displacements of an invariable figure of which the different points describe spherical curves.—The Fourneyron prize for 1903, 1,000 francs, for a theoretical or experimental study of steam turbines.—The Damoiseau prize for 1905, 2,000 francs, for the treatment of the problem: There are some ten comets whose orbits, during their visible periods, are found to be hyperbolic. To investigate, retracing the past and taking account of the perturbations of the planets, whether this was the case before the arrival of these planets in the solar system.—The Binoux prize for 1903, 2,000 francs, for works on the history of the exact sciences.—The d'Ormoy prize, 10,000 francs, for pure and applied mathematics.

THE Royal academy of physical and mathematical sciences, of Naples, announces a prize of 1,000 lire for the best memoir containing a noteworthy contribution to the invariant theory of ternary biquadratic forms, preferably with reference to the various conditions for breaking up into forms of lower order. Memoirs must be submitted by June 30, 1904.

THE Hungarian academy of sciences has issued, under date of February 16, 1903, an official notice of the founding of the Bolyai prize of a medal and 10,000 crowns, to be awarded first in 1905, and every fifth year thereafter, to the author of the best work in mathematics appearing in the preceding five years.

THE annual list of prize questions of the Amsterdam scientific society has recently been issued. Copies of the announcement may be obtained from the secretary of the society, Professor D. J. KORTEWEG, Vondelstraat 104, Amsterdam.

THE several universities below offer during the summer semester of the current academic year courses in mathematics as follows:

UNIVERSITY OF GREIFSWALD. By Professor W. THOMÉ: Potential function, four hours; Theory of algebraic surfaces and space curves, two hours; Mathematical seminar, two hours.

—By Professor E. STUDY: Infinitesimal calculus, four hours; exercises in infinitesimal calculus, one hour; Line geometry, three hours; Mathematical seminar.—By Professor G. KOWALEWSKI: General function theory, four hours; Geometry of numbers, one hour; Exercises in function theory, one hour.

UNIVERSITY OF JENA. By Professor A. GUTZMER: Differential calculus, with exercises, five hours; Introduction to the theory of differential equations, five hours.—By Professor J. THOMAE: Analytic geometry of the plane, four hours; Mathematical geography, four hours.—By Professor G. FREGE: Riemann's theory of functions, four hours; Mathematical exercises, two hours.

AT the University of Nebraska, the following advanced courses in mathematics are offered during the summer session, 1903:—By Professor E. W. DAVIS: Theory of functions.—By Professor R. E. MORITZ: History of mathematics; Mathematical pedagogy.

ON the occasion of the Bolyai celebration at the University of Klausenburg, on January 15, the faculty conferred upon Professor H. POINCARÉ the honorary degree of doctor of philosophy.

DR. H. GRASSMANN has been promoted to a professorship of mathematics at the University of Halle.

PROFESSOR E. VOSS has been called from Würzburg to the University of Munich, as professor of mathematics.

PROFESSOR FRIEDRICH SCHOTTKY, of the University of Berlin, has been elected a member of the Berlin academy of sciences.

PROFESSOR K. HENSEL has succeeded the late Professor L. FUCHS as editor of *Crelle's Journal*.

PROFESSOR G. B. HALSTED, of the University of Texas, has been elected to the chair of mathematics at St. John's College, Annapolis, Md.

DR. M. B. PORTER, of Yale University, has been called to the professorship of mathematics in the University of Texas.

AT Columbia University, Dr. C. J. KEYSER has been promoted to an adjunct professorship of mathematics.

PROFESSOR ALEXANDER JOHNSON has resigned his position

as professor of pure mathematics in McGill University, to take effect at the close of the present academic year.

AT the University of Minnesota, Dr. G. N. BAUER has been promoted to an assistant professorship of mathematics. Dr. G. A. BLISS, instructor in mathematics in the University, is absent on leave, and is spending the year at Göttingen.

PROFESSOR E. J. WILCZYNSKI, of the University of California, has received from the Carnegie Institution an appointment as research assistant and will spend a year in Europe in work on the theory of differential equations and line geometry.

MR. P. A. SMITH has resigned as instructor in mathematics at the University of Illinois to accept a position in the Hiroshina higher normal school, of Japan.

MESSRS. P. H. LINEHAN, C. W. SAXTON, and H. R. FASSETT have been appointed instructors in mathematics in the College of the City of New York.

THE Carnegie Institution has decided to provide for the publication of the works of Dr. G. W. HILL.

PROFESSOR E. L. M. CURTZE died on January 3, 1903, at Thorn, West Prussia, aged 65 years.

REAR Admiral WILLIAM HARKNESS died at New York on February 28, 1903, in his sixty-sixth year. He was for some time astronomical director of the naval observatory, and in 1897 was appointed director of the nautical almanac. In 1893 he was president of the American association for the advancement of science.

MR. W. J. C. MILLER, for many years mathematical editor of *The Educational Times* of London, died on February 11, 1903, at Bristol, England, at the age of seventy-one and a half years. During his connection with the journal mentioned, which covered a period of over thirty years, Mr. Miller published sixty-six volumes of the *Mathematical Reprint*, a series devoted chiefly to problems in modern geometry and higher algebra.

RECENT catalogues of second-hand books on mathematics: K. F. KOEHLER, 6 Kurprinzstrasse, Leipzig, Antiquariats-Katalog no. 557, 2560 titles; no. 559, 27 pages, scientific journals.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CARP (J. A.). Combinatorische configuraties in meerdimensionale ruimten. Utrecht, 1902. 8vo. 68 pp.

CZUBER (E.). Wahrscheinlickeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik und Lebensversicherung. Leipzig, Teubner, 1903. 8vo. 15 + 594 pp. (Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Vol. IX.) M. 24.00

EASTON (B. S.). The constructive development of group theory; with a bibliography. Boston, Ginn, 1902. 8vo. 3 + 89 pp. Cloth. (Publications of the University of Pennsylvania, series in mathematics, No. 2.) $0.75

EMCH (A.). Applications of elliptic functions to problems of closure. 8vo. (*The University of Colorado Studies*, Vol. 1, pp. 81–133.)

GANTER (H.) und RUDIO (F.). Die Elemente der analytischen Geometrie. Zum Gebrauch an höheren Lehranstalten sowie zum Selbststudium. Mit zahlreichen Uebungsbeispielen. Teil I: Die analytische Geometrie der Ebene. 5te, verbesserte Auflage. Leipzig, Teubner, 1903. 8vo. 8 + 187 pp. M. 3.00

GRADHANDT (E.). Beiträge zur Theorie der Fokaleigenschaften der Krümmungskurven auf den Flächen zweiter Ordnung. Rostock, 1901. 8vo. 58 pp.

HASKELL (M. W.). Die Darstellung von gewissen Resultanten in Determinantenform. 8vo. (*Jahresbericht der Deutschen Mathematiker-Vereinigung*, Band 12, pp. 38–42.)

HEDRICK (E. R.). Suplementary note on the calculus of variations. The doctrine of infinity. 8vo. (*Bulletin of the American Mathematical Society* (2) 9, pp. 245–247 and 263–268.)

HELWIG (P. J.). Over een allgemeen gemiddelde en de integralen, die samenhangen met de foutenwet van het meetkundig gemiddelde. Utrecht, 1901. 4to. 79 pp.

HUMBERT (G.). Cours d'analyse, professé à l'Ecole polytechnique. Vol. I. Calcul différentiel; principes du calcul intégral; applications géométriques. Paris, Gauthier-Villars, 1903. 8vo. 484 pp. Fr. 16.00

ISSALY. La géométrie non-euclidienne et l'insuffisance de ses principes. Mémoire faisant suite au Principes fondamentaux de la théorie des pseudo-surfaces, du même auteur. Paris, Hermann, 1902. 8vo. 7 + 62 pp.

KOIALOVICH (B. M.). On a partial differential equation of the fourth order. St. Petersburg, 1902. 8vo. 125 pp. (Russian.)

LÜBSEN (H. B.). Ausführliches Lehrbuch der Analysis zum Selbstunterricht, mit Rücksicht auf die Zwecke des praktischen Lebens bearbeitet. 10te, verbesserte Auflage. Leipzig, Brandstetter, 1902. 8vo. 4 + 203 pp. M. 3.60

NUGTEREN (G. K.). Rationale ruimtekrommen van de 5. ordre. Utrecht, 1901. 8vo. 67 pp.

OVIDIO (E. D'). Geometria analitica. 3a edizione riveduta e corretta. Torino, Bocca, 1903. 8vo. 15 + 520 pp. (Biblioteca matematica, Vol. 8.) Fr. 12.00

RUDIO (F.). See GANTER (H.).

II. ELEMENTARY MATHEMATICS.

AGAPOV (D. V.). Collection of the most common geometrical problems with the methods of their solution. Orenburg, 1902. 8vo. 93 pp., 2 plates. (Russian.) M. 3.50

AMALDI (U.). See ENRIQUES (F.).

BURCH (C.). Short cuts and byways in arithmetic. Full explanations of all principles involved; examples worked out at length. Edinburgh, Blackie, 1902. 8vo. 108 pp. Cloth. 2s.

D. (F. T.). Solutions des exercices et problèmes proposés dans le Cours d'algèbre élémentaire. (Environ 2500 questions.) Lyon and Paris, Vitte, 1902. 16mo. 452 pp.

DIEKMANN. See KOPPE.

ENRIQUES (F.) e AMALDI (U.). Elementi di geometria ad uso delle scuole secondarie superiori. Bologna, Zanichelli, 1902. 16mo. 22 + 655 pp. Fr. 4.50

GREVY (A.). Eléments de géométrie, à l'usage des élèves des classes de seconde et première A et B (programmes du 31 mai 1902). Paris, Nony, 1903. 18mo. 120 pp. Fr. 1.25

——. Eléments de géométrie, à l'usage des élèves des classes de quatrième et troisième A (programmes du 31 mai 1902). Paris, Nony, 1903. 18mo. 172 pp. Fr. 1.50

——. Géométrie à l'usage des élèves des classes de cinquième à troisième B (programmes du 31 mai 1902). Paris, Nony, 1903. 18mo. 306 pp. Fr. 2.25

HIGGS (W. P.). Algebra self-taught; for the use of mechanics, young engineers and home students. 7th edition. New York, Spon and Chamberlin, 1903. 8vo. 104 pp. Cloth. $0.60

HOČEVAR (F.). Lehr- und Uebungsbuch der Geometrie für Untergymnasien. 6te, umgearbeitete Auflage. Leipzig, Freytag, 1902. 8vo. 2 + 122 pp. Cloth. M. 1.50

HUMBERT (E.). Traité d'arithmétique, à l'usage des élèves de mathématiques A et B et des candidats au baccalauréat, avec des compléments destinés aux candidats aux grandes écoles du gouvernement et une préface de J. Tannery. 3e édition. Paris, Nony, 1903. 8vo. 7 + 501 pp. Fr. 5.00

INGRAMI (G.). Complementi di matematica ad uso del liceo. Bologna, Cuppini, 1902. 8vo. 119 pp. Fr. 2.00

JACQUET (E.) et LACLEF (A.). Arithmétique du brevet élémentaire et des cours complémentaires, suivie d'un complément de géométrie pratique (490 exercices et problèmes). Nouvelle édition, revue et corrigée. Paris, Nathan, 1903. 12mo. 6 + 292 pp. Fr. 1.75

KOPPE und DIEKMANN. Geometrie. Teil 1: Ausgabe für Gymnasien. 18te Auflage. Ergänzungsheft. Essen, Baedeker, 1903. 8vo. 62 pp. M. 0.60

LACLEF (A.). See JACQUET (E.).

MARTINI ZUCCAGNI (A.). Guida per la risoluzione degli esercizi d'algebra. Livorno, Giusti, 1902. 16mo. 6 + 131 pp. Fr. 1.00

MATTHIESSEN (L.). Kommentar zur Sammlung von Beispielen und Aufgaben aus der allgemeinen Arithmetik und Algebra von E. Heis. Für die Schüler von Gymnasien, Realgymnasien, Oberrealschulen und Gewerbschulen bearbeitet. 4te, verbesserte Auflage. Köln, Du Mont-Schauberg, 1902. 8vo. 8 + 180 pp. M. 2.50

MOČNIK (F. VON). Lehrbuch der Arithmetik für Untergymnasien, bearbeitet von A. Neumann. 1te Abteilung, für die I. und II. Klasse. 36ste, veränderte Auflage. Leipzig, Freytag, 1902. 8vo. 3 + 148 pp. Cloth. M. 2.00

MODEL ANSWERS to intermediate pure mathematics: being the mathematical papers set at the London University intermediate examinations in arts and science, from 1895 to 1902. (*Univ. Tut. Series.*) London, Clive, 1903. 8vo. 146 pp. 2s. 6d.

NEUMANN (A.). See MOČNIK (F. VON).

ROBERTS (R.). A new geometry for beginners, theoretical and practical. Edinburgh, Blackie, 1903. 8vo. 88 pp. Cloth. 1s. 6d.

SALOMON (A.). Leçons de géométrie, à l'usage de l'enseignement secondaire des jeunes filles. Géométrie dans l'espace. (Classe de cinquième année.) Paris, Nony, 1903. 18mo. 129 pp. Fr. 1.25

SCHMID (A.). See SICKENBERGER (A.).

SENDLER (R.). Raumlehre für Präparandenanstalten. Nach dem Lehrplane vom 1. VII. 1901 bearbeitet. 6te Auflage. Breslau, Handel, 1903. 8vo. 8 + 147 pp. M. 1.60

SICKENBERGER (A.). Uebungsbuch zur Algebra. Bearbeitet von A. Schmid. Abteilung I: 1. und 2. Stufe der Rechnungsarten einschliesslich der lineären Gleichungen mit einer und mehreren Unbekannten. 4te Auflage. 5 + 106 pp. Abteilung II: 3. Stufe der Rechnungsarten, quadratische Gleichungen, Reihen. 3te, vermehrte Auflage. 3 + 128 pp. München, Ackermann, 1903. 8vo. M. 3.00

STOOPS (W. H.). A general method for the geometrical trisection of angles and arcs, with accompanying diagrams; supplemented with a formal proof and a trigonometrical analysis. Riverside, N. J., Stoops, [1902]. 4to. 7 + 34 pp. Cloth. $1.75

TANNERY (J.). See HUMBERT (E.).

TESTI (G. M.). Sulle combinazioni con repetizione di m elementi n ad n. Livorno, Giusti, 1902. 8vo. 4 pp.

THIEME (H.). Leitfaden der Mathematik für Realanstalten. Teil 2: Die Oberstufe. Leipzig, Freytag, 1902. 8vo. 4 + 196 pp. Cloth. M. 2.50

VECCHI (L.). Primi elementi d'algebra esposti ad uso dei licenziandi delle scuole tecniche, sulle norme dei vigenti programmi ministeriali. Cremona, Tipografia Sociale, 1902. 8vo. 43 pp.

WENTWORTH (G. A.). Plane trigonometry. 2d revised edition. Boston, Ginn, 1903. 12mo. 6 + 141 + 21 pp. Cloth.

III. APPLIED MATHEMATICS.

BOCCARDI (I.). Guide du calculateur. Astronomie, géodésie, navigation. 2 vols. Paris, 1902. 4to. 10 + 80 + 4 + 140 pp. Fr. 16.00

DAYMARD (M.). Mémoire sur de nouvelles courbes servant à représenter et à mesurer la stabilité statique des navires sous toutes les inclinaisons possibles. Paris, Bernard, 1902. 4to. 32 pp.

DUHEM (P.). Thermodynamique et chimie. Paris, 1902. 8vo. 500 pp. Fr. 15.00

HEIMANN (H.). Ueber die äquivalenten Pole eines Magnets und die Methoden zu ihrer Berechnung. (Diss.) Rostock, 1902. 8vo. 55 pp., 1 plate. M. 1.80

HOROVITZ (G.). Ueber die Wärmeausnützung in der Gasmaschine. (Diss.) Berlin, 1902. 4to. 32 pp. M. 2.00

JACOANGELI (O.). Teoria nuova e generale della divisione dei terreni per terre di equale e diversa valenza. Torino, Bona, 1902. 8vo. 86 pp., 2 plates.

KIRPICHOV (V. S.). Elements of graphical statics. Kiev, 1902. 8vo. 268 pp., 74 plates. (Russian.) M. 8.50

LANDRÉ (C. L.). See SCHOUTEN (P.).

LIPPMANN (O.). Flächenberechnungen (Planimetrie), Körperberechnungen (Stereometrie) und Gewichtsberechnungen mit besonderer Berücksichtigung des Maschinenbaues. Lehrgang mit 198 ausgerechneten Beispielen, 95 Formeln, 13 Tabellen und Anleitungen zum Gebrauche derselben. Dresden, Höckner, [1902]. 8vo. 8 + 114 pp. M. 1.50

MEITZEN (A.). Geschichte, Theorie und Technik der Statistik. 2te Auflage. Stuttgart, 1903. 8vo. 10 + 240 pp. M. 6.00

PERRON (O.). Ueber die Drehung eines starren Körpers um seinen Schwerpunkt bei Wirkung äusserer Kräfte. München, 1902. 43 pp.

REACH (T. C. F.). See SCHOUTEN (P.).

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